BY GIMUSI 01/11/15

Limiti 14

Argomenti: limiti di funzioni e successioni **Difficoltà:** ⋆⋆⋆⋆⋆

Prerequisiti: tutte le tecniche per il calcolo di limiti

Calcolare i seguenti limiti di funzione.

	a) Funzione	Limite	b) Successione	Limite
1)	$\lim_{x\to+\infty}\left(x^2\sin\frac{1}{x}-\frac{x^2}{x+1}\right)$	1	$\lim_{x\to+\infty} e^x\left(\log(e^x+1)-x\right)$	1
2)	$\lim_{x\to0}\frac{\cos(\arctan x)-\cosh(\arctan x)}{\sinh^2(\sqrt{x})\cdot\tan^2(\sqrt{x})}$	-1	$\lim_{x\to0}\left[1-\left(\frac{\sin x}{x}\right)^{x^2}\right]\cdot\frac{1}{x^4}$	1/6
3)	$\lim_{x\to0}\frac{\arctan(\cos x)-\arctan(\cosh x)}{\sinh^2(\sqrt{x})\cdot\tan^2(\sqrt{x})}$	-1/2	$\lim_{x\to0}\frac{\arctan(x^5)-\arctan^5 x}{x^7}$	5/3
4)	$\lim_{x\to0}\frac{\cos(\cosh x)-\cos(\sqrt{1+x^2})}{\cosh(\cos x)-\cosh(\sqrt{1-x^2})}$	$-\frac{\sin(1)}{\sinh(1)}$	$\lim_{x\to0}\left[\tan\left(\frac{\arccos x}{2}\right)\right]^{1/x}$	1/e

Calcolare i limiti delle seguenti successioni.

	a) Successione	Limite	b) Successione	Limite
5)	$\sin^n(\cos(n!+3^n))$	0	$\frac{n^{n^e}-e^{e^n}}{e^{n^e}-n^{e^n}}$	0
6)	$n^2\left(\arctan(n+1)-\arctan n\right)$	1	$n\left(\sin\frac{n+5}{n+1}-\sin\frac{n+1}{n+5}\right)$	8cos(1)
7)	$\left[\sqrt{n+1}+\sqrt{4n+1}-3\sqrt{n}\right]^{1/\log n}$	$e^{-1/2}$	$\frac{\sqrt[n]{(n+1)!}-\sqrt[n]{n!}}{\log n}$	1/e
8)	$n\left(\arcsin\frac{n-1}{n}-\arcsin\frac{n-2}{n}\right)$	$+\infty$	$\sqrt{n}\left[\sqrt{\pi}-\sqrt{\arccos\frac{1-n}{n}}\right]$	$\frac{1}{\sqrt{2\pi}}$
9)	$n^2\left(\sqrt[4]{\frac{2n^2+3}{n^2+1}}-\sqrt[4]{\frac{2n^3+3}{n^3+1}}\right)$	$\frac{\sqrt[4]{2}}{8}$	$n\left[\sqrt[n]{\binom{3n}{n+1}}-\sqrt[n]{\binom{3n}{n}}\right]$	$\frac{27\lg 2}{4}$
10)	$n^4\left(\sqrt[4]{\frac{16n^2-8}{n^2+2}}-\sqrt[3]{\frac{8n^2+1}{n^2+2}}\right)$	$-\frac{25}{64}$	$n\sin\left(\frac{n!+1}{n}\pi\right)$	π

[to be completed, tanto alla cattiveria non c'e' limite]

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1.a) $\lim_{x\to+\infty}\left(x^2\sin\frac{1}{x}-\frac{x^2}{x+1}\right)=1$

$$x^2\sin\frac{1}{x}-\frac{x^2}{x+1}=x^2\left(\frac{1}{x}-\frac{1}{6x^3}+o\left(\frac{1}{x^3}\right)\right)-\frac{x^2}{x+1}=$$

$$=x-\frac{1}{6x}-\frac{x^2}{x+1}+o\left(\frac{1}{x}\right)=\frac{\cancel{6x^3}+6x^2-x-1-\cancel{6x^3}}{6x(x+1)}+o\left(\frac{1}{x}\right)\to 1$$

1.b) $\lim_{x\to+\infty}e^x(\log(e^x+1)-x)=1$

$$e^x(\log(e^x+1)-x)=e^x(\log(e^x+1)-\log e^x)=$$

$$=e^x\log\left(1+\frac{1}{e^x}\right)=e^x\left(\frac{1}{e^x}+o\left(\frac{1}{e^x}\right)\right)\to 1$$

2.a) $\lim_{x\to 0}\frac{\cos(\arctan x)-\cosh(\arctan x)}{\sin^2(\sqrt{x})\cdot\tan^2(\sqrt{x})}=-1$

$$\frac{\cos(\arctan x)-\cosh(\arctan x)}{\sin^2(\sqrt{x})\cdot\tan^2(\sqrt{x})}=\frac{\cos(x+o(x))-\cosh(x+o(x))}{(\sqrt{x}+o(\sqrt{x}))^2+(\sqrt{x}+o(\sqrt{x}))^2}=$$

$$=\frac{1-x^2/2-1-x^2/2+o(x^2)}{(x+o(x))(x+o(x))}=\frac{-x^2+o(x^2)}{x^2+o(x^2)}\to -1$$

2.b) $\lim_{x\to 0}\left[1-\left(\frac{\sin x}{x}\right)^{x^2}\right]\frac{1}{x^5}=1/6$

$$\left[1-\left(\frac{\sin x}{x}\right)^{x^2}\right]\frac{1}{x^5}=\left[1-e^{x^2\log\left(1-x^2/6+o(x^2)\right)}\right]\frac{1}{x^5}=$$

$$=\left[1-e^{x^2\left(-\frac{x^2}{6}+o(x^2)\right)}\right]\frac{1}{x^5}=\frac{x^5/6+o(x^5)}{x^5}\to 1/6$$

3.a) $$\lim_{x\to 0} \frac{\arctan(\cos x)-\arctan(\cosh x)}{\sinh^2(\sqrt{x})\cdot\tan^2(\sqrt{x})} = -1/2$$

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$$\arctan(\cos x)-\arctan(\cosh x) \overset{\downarrow}{=} \arctan\left(\frac{\cos x-\cosh x}{1+\cos x\cdot\cosh x}\right) = -\frac{x^2}{2}+o(x^2)$$

$$\frac{\cos x-\cosh x}{1+\cos x\cdot\cosh x} = \left(1-x^2/2-1-x^2/2+o(x^2)\right)\cdot\left(2+o(x^2)\right)^{-1} =$$

$$= \frac{1}{2}\left(-x^2+o(x^2)\right)\left(1+o(x^2)\right) = -\frac{x^2}{2}+o(x^2)$$

$$\sinh^2(\sqrt{x})\cdot\tan^2(\sqrt{x}) = \left(\sqrt{x}+o(\sqrt{x})\right)^2\left(\sqrt{x}+o(\sqrt{x})\right)^2 =$$

$$= \left(x+o(x)\right)\left(x+o(x)\right) = x^2+o(x^2)$$

$$\frac{\arctan(\cos x)-\arctan(\cosh x)}{\sinh^2(\sqrt{x})\cdot\tan^2(\sqrt{x})} = \frac{-\frac{x^2}{2}+o(x^2)}{x^2+o(x^2)} = -\frac{1}{2}$$

3.b) $$\lim_{x\to 0} \frac{\arctan(x^5)-\arctan^5 x}{x^7} = 5/3$$

$$\arctan(x^5) = x^5+o(x^{15})$$

$$\arctan^5(x) = \left(x-x^3/3+x^5/5+o(x^5)\right)^5 = x^5-5x^7/3+o(x^7)$$

$$\frac{\arctan(x^5)-\arctan^5 x}{x^7} = \frac{x^5-x^5+5x^7/3+o(x^7)}{x^7} \to \frac{5}{3}$$

5.a) $$\lim_{x\to 0} \frac{\cos(\cosh x)-\cos(\sqrt{1+x^2})}{\cosh(\cos x)-\cosh(\sqrt{1-x^2})} = \frac{-\sin(1)}{\sinh(1)}$$

$$\cosh x = 1+x^2/2+x^4/24+o(x^4)$$

$$\sqrt{1+x^2} = 1+x^2/2-x^4/8+o(x^4)$$

$$\cos(\cosh x)-\cos(\sqrt{1+x^2}) = -2\sin\left(\frac{\cosh x+\sqrt{1+x^2}}{2}\right)\sin\left(\frac{x^4}{12}+o(x^4)\right) =$$

$$= -2\,\sin\left(\frac{\cosh x + \sqrt{1+x^2}}{2}\right)\left(\frac{x^5}{12} + o(x^5)\right)$$

$$\cos x = 1 - x^2/2 + x^4/24 + o(x^5)$$

$$\sqrt{1-x^2} = 1 - x^2/2 - x^4/8 + o(x^5)$$

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$$\cosh(\cos x) - \cosh(\sqrt{1-x^2}) \overset{\downarrow}{=} 2\sinh\left(\frac{\cos x + \sqrt{1-x^2}}{2}\right)\sinh\left(\frac{x^5}{12} + o(x^5)\right) =$$

$$= 2\sinh\left(\frac{\cos x + \sqrt{1-x^2}}{2}\right)\left(\frac{x^5}{12} + o(x^5)\right)$$

$$\frac{\cos(\cosh x) - \cos(\sqrt{1+x^2})}{\cosh(\cos x) - \cosh(\sqrt{1-x^2})} = \frac{-\cancel{2}\,\sin\overbrace{\left(\frac{\cosh x + \sqrt{1+x^2}}{2}\right)}^{\to 1}\left(\frac{x^5}{12} + o(x^5)\right)}{\cancel{2}\sinh\underbrace{\left(\frac{\cos x + \sqrt{1-x^2}}{2}\right)}_{\to 1}\left(\frac{x^5}{12} + o(x^5)\right)} \to$$

$$\to \frac{-\sin(1)}{\sinh(1)}$$

5.B) $\lim_{x \to 0} \left[\tan\left(\frac{\arccos x}{2}\right)\right]^{1/x} = 1/e$

$$\tan\left(\frac{\arccos x}{2}\right) = \sqrt{\frac{1-\cos(\arccos x)}{1+\cos(\arccos x)}} = \sqrt{\frac{1-x}{1+x}} =$$

$$= \left[(1-x)(1-x+o(x)\right]^{1/2} = (1-2x+o(x))^{1/2} = 1 - x + o(x)$$

$$\left[\tan\left(\frac{\arccos x}{2}\right)\right]^{1/x} = e^{\frac{1}{x}\log(1-x+o(x))} = e^{-1+o(1)} \to e^{-1}$$

5.a) $\sin^n(\cos(n!+3^n)) \to 0$

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$$-1 \le \cos(n!+3^n) \le 1 \;\overset{\downarrow}{\Rightarrow}\; \sin(-1) \le \sin(\cos(n!+3^n)) \le \sin(1)$$

$$\Rightarrow \quad 0 \le |\overset{\to 0}{\sin^n}(\cos(n!+3^n))| \le |\overset{\to 0}{\sin^n}(1)|$$

$$\frac{n^{n^e}-e^{e^n}}{e^{n^e}-n^{e^n}}$$

5.b) $\dfrac{n^{n^e}-e^{e^n}}{e^{n^e}-n^{e^n}} \to 0$

$$\frac{n^{n^e}-e^{e^n}}{e^{n^e}-n^{e^n}} = \frac{\overset{\to 0}{\dfrac{n^{n^e}}{n^{e^n}}} - \overset{\to 0}{\dfrac{e^{e^n}}{n^{e^n}}}}{\underset{\to 0}{\dfrac{e^{n^e}}{n^{e^n}}} - 1} \to 0$$

6.a) $n^2(\arctan(n+1)-\arctan(n)) \to 1$

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$$\arctan(n+1)-\arctan(n) \overset{\downarrow}{=} \arctan\left(\frac{1}{1+n(n+1)}\right) = \frac{1}{n(n+1)} + o\left(\frac{1}{n^2}\right)$$

$$\frac{1}{1+n(n+1)} = \frac{\frac{1}{n(n+1)}}{\frac{1}{n(n+1)}+1} = \frac{1}{n(n+1)}\left(1+o\left(\frac{1}{n^2}\right)\right) = \frac{1}{n(n+1)}+o\left(\frac{1}{n^2}\right)$$

$$n^2(\arctan(n+1)-\arctan(n)) = \frac{n^2}{n(n+1)} + o(1) \to 1$$

6.b) $n\left(\sin\frac{n+5}{n+1} - \sin\frac{n+1}{n+5}\right) \to 8\cos(1)$

$$\sin\frac{n+5}{n+1} - \sin\frac{n+1}{n+5} = 2\cos\left[\frac{1}{2}\left(\frac{n+5}{n+1}+\frac{n+1}{n+5}\right)\right]\sin\left[\frac{1}{2}\left(\frac{n+5}{n+1}-\frac{n+1}{n+5}\right)\right]$$

$$\frac{n+5}{n+1}-\frac{n+1}{n+5} = \frac{\cancel{n^2}+10n+25-\cancel{n^2}-2n-1}{(n+1)(n+5)} = \frac{8n+24}{n^2+6n+5} =$$

$$= (8n+25)\cdot\frac{1}{n^2}\left(1+6/n+5/n^2\right)^{-1} = \frac{(8n+25)}{n^2}\left(1+o(1)\right) = \frac{8}{n}+o\left(\frac{1}{n}\right)$$

$$\sin\left[\frac{1}{2}\left(\frac{n+5}{n+1}-\frac{n+1}{n+5}\right)\right] = \sin\left(\frac{4}{n}+o\left(\frac{1}{n}\right)\right) = \frac{4}{n}+o\left(\frac{1}{n}\right)$$

$$n\left(\sin\frac{n+5}{n+1}-\sin\frac{n+1}{n+5}\right) = n\cdot 2\cos\underbrace{\left[\frac{1}{2}\left(\frac{n+5}{n+1}+\frac{n+1}{n+5}\right)\right]}_{\to 1}\left(\frac{4}{n}+o\left(\frac{1}{n}\right)\right) \to 8\cos(1)$$

7.a) $$\left(\sqrt{n+1}+\sqrt{5n+1}-3\sqrt{n}\right)^{1/\log n} \to e^{-1/2}$$

$$\sqrt{n+1}+\sqrt{5n+1}-3\sqrt{n} = \sqrt{n}\left(\left(1+1/n\right)^{1/2}+2\left(1+1/5n\right)^{1/2}-3\right) =$$

$$= \sqrt{n}\left(\cancel{1}+1/2n+\cancel{2}+1/5n-\cancel{3}+o(1/n)\right) = \sqrt{n}\left(3/5n+o(1/n)\right) =$$

$$= \frac{3}{5}\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)$$

$$\left(\sqrt{n+1}+\sqrt{5n+1}-3\sqrt{n}\right)^{1/\log n} = e^{\frac{\log\left(\frac{3}{5}\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right)}{\log n}} =$$

$$= e^{\frac{\log\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right)+\log\left(\frac{3}{5}\frac{1}{\sqrt{n}}\right)}{\log n}} = e^{\frac{\log\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right)-\frac{1}{2}\log n+\log\frac{3}{5}}{\log n}} \to e^{-1/2}$$

7.b) $$\frac{\sqrt[n]{(n+1)!}-\sqrt[n]{n!}}{\log n} \to 1/e$$

$$\frac{\sqrt[n]{(n+1)!}-\sqrt[n]{n!}}{\log n} = \sqrt[n]{n!}\,\frac{\left(\sqrt[n]{(n+1)}-1\right)}{\log n}$$

$$\sqrt[n]{(n+1)} = e^{\frac{1}{n}\log(n+1)} = 1+\frac{\log(n+1)}{n}+\frac{\log^2(n+1)}{2n^2}+o\left(\frac{\log^2 n}{n^2}\right)$$

$$\frac{\left(\sqrt[n]{(n+1)}-1\right)}{\log n} = \frac{\log(n+1)}{n\log n}+\frac{\log^2(n+1)}{2n^2\log n}+o\left(\frac{\log n}{n^2}\right)$$

$$\sqrt[n]{n!}\,\frac{\sqrt[n]{(n+1)}-1}{\log n} = \sqrt[n]{n!}\left(\frac{\log(n+1)}{n\log n}+\frac{\log^2(n+1)}{2n^2\log n}+o\left(\frac{\log n}{n^2}\right)\right)=$$

$$=\overset{\to 1/e}{\frac{\sqrt[n]{n!}}{n}}\left(\overset{\to 1}{\frac{\log(n+1)}{\log n}}+\overset{\to 0}{\frac{\log^2(n+1)}{2n\log n}}+o\overset{\to 0}{\left(\frac{\log n}{n}\right)}\right)\to 1/e$$

8. a) $n\left(\arcsin\left(\frac{n-1}{n}\right)-\arcsin\left(\frac{n-2}{n}\right)\right)\to +\infty$

$$\sin(a-b)=\sin a\cos b-\sin b\cos a$$

$$\sin\left(\arcsin\left(\frac{n-1}{n}\right)-\arcsin\left(\frac{n-2}{n}\right)\right)=\frac{n-1}{n}\sqrt{1-\left(\frac{n-2}{n}\right)^2}-\frac{n-2}{n}\sqrt{1-\left(\frac{n-1}{n}\right)^2}$$

$$\arcsin\left(\frac{n-1}{n}\right)-\arcsin\left(\frac{n-2}{n}\right)=\arcsin\left(\frac{n-1}{n}\sqrt{1-\left(\frac{n-2}{n}\right)^2}-\frac{n-2}{n}\sqrt{1-\left(\frac{n-1}{n}\right)^2}\right)$$

$$\sqrt{1-\left(\frac{n-2}{n}\right)^2}=\left(1-\frac{n^2-4n+4}{n^2}\right)^{1/2}=\left(\cancel{1}-\cancel{1}+\frac{4}{n}-\frac{4}{n^2}\right)^{1/2}=$$

$$=\frac{2}{\sqrt{n}}\left(1-4/n\right)^{1/2}=\frac{2}{\sqrt{n}}\left(1-\frac{2}{n}+o\left(\frac{1}{n}\right)\right)=\frac{2}{\sqrt{n}}+o\left(\frac{1}{n}\right)$$

$$\sqrt{1-\left(\frac{n-1}{n}\right)^2}=\left(1-\frac{n^2-2n+1}{n^2}\right)^{1/2}=\left(\cancel{1}-\cancel{1}+\frac{2}{n}-\frac{1}{n^2}\right)^{1/2}=\frac{\sqrt{2}}{\sqrt{n}}+o\left(\frac{1}{n}\right)$$

$$\frac{n-1}{n}\sqrt{1-\left(\frac{n-2}{n}\right)^2}-\frac{n-2}{n}\sqrt{1-\left(\frac{n-1}{n}\right)^2}=\frac{2(n-1)}{n\sqrt{n}}-\frac{\sqrt{2}(n-2)}{n\sqrt{n}}+o\left(\frac{1}{n}\right)=$$

$$=\frac{2-\sqrt{2}}{\sqrt{n}}+o\left(\frac{1}{n}\right)$$

$$\arcsin\left(\frac{n-1}{n}\sqrt{1-\left(\frac{n-2}{n}\right)^2}-\frac{n-2}{n}\sqrt{1-\left(\frac{n-1}{n}\right)^2}\right)=\frac{2-\sqrt{2}}{\sqrt{n}}+o\left(\frac{1}{n}\right)$$

$$n\arcsin\left(\frac{n-1}{n}\sqrt{1-\left(\frac{n-2}{n}\right)^2}-\frac{n-2}{n}\sqrt{1-\left(\frac{n-1}{n}\right)^2}\right)=n\frac{2-\sqrt{2}}{\sqrt{n}}+o(1)\to+\infty$$

8.b) $\sqrt{n}\left[\sqrt{\pi}-\sqrt{\arccos\frac{1-n}{n}}\right] \to \frac{1}{\sqrt{2\pi}}$

$$\arccos\frac{1-n}{n} = ? \qquad \arccos(x-1) \quad x\to 0^+$$

$$(\arccos(x-1))' = -\frac{1}{\sqrt{1-(x-1)^2}} = -\frac{1}{\sqrt{2x-x^2}} = -\frac{1}{\sqrt{2x}}(1-x/2)^{-1/2} =$$

$$= -\frac{1}{\sqrt{2x}}(1+x/4+o(x)) = -\frac{1}{\sqrt{2x}} - \frac{\sqrt{x}}{4\sqrt{2}} + o(\sqrt{x})$$

$$\arccos(x-1) = \int(\arccos(x-1))' = \pi - \sqrt{2x} - \frac{x^{3/2}}{6\sqrt{2}} + o(x^{3/2})$$

$$\sqrt{\arccos(x-1)} = (\pi-\sqrt{2x}+o(\sqrt{x}))^{1/2} = \sqrt{\pi}(1-\sqrt{2x}/\pi+o\sqrt{x})^{1/2} =$$

$$= \sqrt{\pi}(1-\sqrt{2x}/2\pi + o\sqrt{x}) = \sqrt{\pi} - \sqrt{2\pi x}/2\pi + o(x)$$

$$\sqrt{\arccos\frac{1-n}{n}} = \sqrt{\arccos\left(\frac{1}{n}-1\right)} = \sqrt{\pi} - \frac{1}{\sqrt{2\pi}\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

$$\sqrt{n}\left[\sqrt{\pi}-\sqrt{\arccos\frac{1-n}{n}}\right] = \frac{1}{\sqrt{2\pi}} + o(1) \to \frac{1}{\sqrt{2\pi}}$$

9.a) $n^2\left(\sqrt[5]{\frac{2n^2+3}{n^2+1}} - \sqrt[5]{\frac{2n^3+3}{n^3+1}}\right) \to \frac{\sqrt[5]{2}}{8}$

$$\sqrt[5]{\frac{2n^2+3}{n^2+1}} = (2n^2+3)^{1/5}(n^2+1)^{-1/5} =$$

$$= (2n^2)^{1/5}(1+3/2n^2)^{1/5}(n^2)^{-1/5}(1+1/n^2)^{-1/5} =$$

$$= \sqrt[5]{2}\,(1+3/8n^2+o(1/n^2))(1-1/5n^2+o(1/n^2)) =$$

$$= \sqrt[5]{2}\,(1+1/8n^2+o(1/n^2))$$

$$\sqrt[5]{\frac{2n^3+3}{n^3+1}} = (2n^3+3)^{1/5}\,(n^3+1)^{-1/5} =$$

$$=(2n^3)^{1/5}(1+3/2n^3)^{1/5}\,(n^3)^{-1/5}(1+1/n^3)^{-1/5} =$$

$$= \sqrt[5]{2}\,\left(1+3/8n^3+o(1/n^3)\right)\left(1-1/5n^3+o(1/n^3)\right) =$$

$$= \sqrt[5]{2}\,\left(1+1/8n^3+o(1/n^3)\right)$$

$$\sqrt[5]{\frac{2n^2+3}{n^2+1}} - \sqrt[5]{\frac{2n^3+3}{n^3+1}} = \frac{\sqrt[5]{2}}{8n^2} + o\left(\frac{1}{n^2}\right)$$

$$n^2\left(\sqrt[5]{\frac{2n^2+3}{n^2+1}} - \sqrt[5]{\frac{2n^3+3}{n^3+1}}\right) = \frac{\sqrt[5]{2}}{8} + o(1) \longrightarrow \frac{\sqrt[5]{2}}{8}$$

9.b) $n\left[\sqrt[n]{\binom{3n}{n+1}} - \sqrt[n]{\binom{3n}{n}}\right] \longrightarrow \frac{27}{5}\log 2$

$$\sqrt[n]{\binom{3n}{n+1}} - \sqrt[n]{\binom{3n}{n}} = \sqrt[n]{\binom{3n}{n}}\left(\frac{\sqrt[n]{\binom{3n}{n+1}}}{\sqrt[n]{\binom{3n}{n}}} - 1\right)$$

$$\frac{\sqrt[n]{\binom{3n}{n+1}}}{\sqrt[n]{\binom{3n}{n}}} = \left(\binom{3n}{n+1}\Big/\binom{3n}{n}\right)^{1/n} = \left(\frac{\cancel{(3n)!}}{(n+1)\cancel{!}\,\cancel{(2n-1)!}}\;\frac{\cancel{n!}\,(2n)\cancel{!}}{\cancel{(3n)!}}\right)^{1/n} =$$

$$= \left(\frac{2n}{n+1}\right)^{1/n} = \left(2\,(1+1/n)^{-1}\right)^{1/n} = \left(2-2/n+o(1/n)\right)^{1/n} =$$

$$= e^{\frac{1}{n}\left(\log 2 + \log(1-1/n+o(1/n))\right)} =$$

$$= e^{\frac{1}{n}\left(\log 2 - 1/n + o(1/n)\right)} = e^{\frac{\log 2}{n} + o(1/n)} =$$

$$= 1 + \frac{\log 2}{n} + o(1/n)$$

$$\sqrt[n]{\binom{3n}{n+1}}-\sqrt[n]{\binom{3n}{n}}=\sqrt[n]{\binom{3n}{n}}\left(\frac{\log 2}{n}+o(1/n)\right)$$

$$n\left[\sqrt[n]{\binom{3n}{n+1}}-\underbrace{\sqrt[n]{\binom{3n}{n}}}_{\to\frac{27}{4}}\right]=\sqrt[n]{\binom{3n}{n}}\left(\log 2+o(1)\right)\to\frac{27}{4}\log 2$$

$\sqrt[n]{\binom{3n}{n}}\to\frac{27}{4}$ (CRIT. RAPP.-RADICE) VD. SCHEDA 24

$$b_n=\binom{3n}{n}\qquad\frac{b_{n+1}}{b_n}=\frac{(3n+3)!}{(n+1)!(2n+2)!}\,\frac{n!\,(2n)!}{(3n)!}=\frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)}\to\frac{27}{4}$$

10.a) $n^4\left(\sqrt[4]{\dfrac{16n^2-8}{n^2+2}}-\sqrt[3]{\dfrac{8n^2+1}{n^2+2}}\right)\to-\dfrac{25}{64}$

$$\sqrt[4]{\frac{16n^2-8}{n^2+2}}=(16n^2)^{1/4}(1-1/2n^2)^{1/4}(n^2)^{-1/4}(1+2/n^2)^{-1/4}=$$

$$=2\left(1-1/8n^2-3/128n^4+o(1/n^4)\right)\left(1-1/2n^2+5/8n^4+o(1/n^4)\right)=$$

$(8-3+80)/128=85/128$

$$=2\left(1-5/8n^2+(1/16-3/128+5/8)/n^4+o(1/n^4)\right)$$

$$=2-5/4n^2+85/64n^4+o(1/n^4)$$

$$\sqrt[3]{\frac{8n^2+1}{n^2+2}}=(8n^2)^{1/3}(1+1/8n^2)^{1/3}(n^2)^{-1/3}(1+2/n^2)^{-1/3}=$$

$$=2\left(1+1/24n^2-(1/9)(1/64n^4)+o(1/n^4)\right)\left(1-2/3n^2+8/9n^4+o(1/n^4)\right)=$$

$(-16-1+512)/576=495/576=55/64$

$$=2\left(1-5/8n^2+(-1/36-1/576+8/9)/n^4+o(1/n^4)\right)$$

$$=2-5/4n^2+55/32n^4+o(1/n^4)$$

$$n^4\left(\sqrt[4]{\frac{16n^2-8}{n^2+2}}-\sqrt[3]{\frac{8n^2+1}{n^2+2}}\right)=\frac{85}{64}-\frac{55}{32}+o(1)\to-\frac{25}{64}$$

10.b) $n \sin\left(\frac{n!+1}{n}\pi\right) \to \pi$

INTERO $k(n)$

$$\frac{n!+1}{n}\pi = (n-1)!\,\pi + \frac{\pi}{n} = \frac{(n-1)!}{2}\,2\pi + \frac{\pi}{n} = 2k(n)\pi + \frac{\pi}{n}$$

$$\sin\left(\frac{n!+1}{n}\pi\right) = \sin\left(2k(n)\pi + \frac{\pi}{n}\right) = \sin(\pi/n) = \pi/n + o(1/n)$$

$$n \sin\left(\frac{n!+1}{n}\pi\right) = \pi + o(1) \to \pi$$