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Limiti 10

Argomenti: limiti di funzioni **Difficoltà:** ⋆⋆⋆⋆

Prerequisiti: limiti notevoli, confronto di ordini di infinito e di infinitesimo

In ogni riga è assegnata una funzione, di cui si chiede di calcolare il limite per x tendente a ciascuno dei valori indicati (se la richiesta non ha senso, accorgersene e segnalarlo).

	Funzione	a) $x \to$	Limite	b) $x \to$	Limite	c) $x \to$	Limite	d) $x \to$	Limite
1)	$\dfrac{\sin(\log x)}{\log x}$	0^+	0	1	1	e	SIN1	$+\infty$	0
2)	$\dfrac{\log(x^3+1)}{\log(x^2+1)}$	$-\infty$	—	0	0	1	1	$+\infty$	3/2
3)	$\dfrac{\sin x}{x-\pi}$	$-\infty$	0	0^+	0	π	−1	$+\infty$	0
4)	$\dfrac{\cos x}{(2x-\pi)^2}$	0	$1/\pi^2$	$\pi/2$	—	$3\pi/2$	0	$+\infty$	0
5)	$\dfrac{x^2+x-2}{\sqrt{x}-1}$	0^-	2	0^+	2	1	6	$+\infty$	$+\infty$
6)	$\dfrac{\log(x^2-3)}{x^4-x-14}$	$-\infty$	0	$(\sqrt{3})^+$	$+\infty$	2	5/31	$+\infty$	0
7)	$(x^2+3)^{1/\log x}$	0^+	1	1^-	0	1^+	$+\infty$	$+\infty$	e^2
8)	$(\sin x)^{\tan^2 x}$	0^+	1	$(\pi/2)^+$	$1/\sqrt{e}$	$(\pi/2)^-$	$1/\sqrt{e}$	$+\infty$	—
9)	$(\tan x)^{\cos x}$	0^+	0	$\pi/4$	1	$(\pi/2)^-$	1	$-\pi^-$	$-\infty$

Calcolare i limiti delle seguenti successioni.

	a) Successione	Limite	b) Successione	Limite	c) Successione	Limite
10)	$\dfrac{(\log n)^n}{n^4 \cdot 4^n}$	$+\infty$	$\dfrac{\log(n\log n)}{\log(n\log^3 n)}$	1	$\dfrac{\sqrt{n+\sqrt{n^3+2}}}{\sqrt[4]{n^3-7}}$	1
11)	$\dfrac{27^{\sqrt[3]{n}}}{3^{\sqrt{n}}}$	0	$\dfrac{\log(1+3^n)}{\log(1+2^n)}$	$\dfrac{\log 3}{\log 2}$	$(\log n)^n - n^{\log n}$	$+\infty$
12)	$\sqrt[n]{\sqrt[n]{n!}}$	1	$\sqrt[n]{\sqrt[n]{n}-\sqrt[n]{2}}$	1	$\sqrt[2n]{n!}\cdot \arcsin\dfrac{\sqrt{n+2}}{n}$	0
13)	$\left(\dfrac{n}{n+6}\right)^n$	e^{-6}	$\dfrac{n^{\sqrt{n}}}{(\log n)^{\log n}}$	$+\infty$	$\dfrac{(n+3)^n+3^n}{(n+7)^n+10^n}$	e^{-5}

1) $\dfrac{\sin(\log x)}{\log x}$

(a) $x \to 0^+$ $\quad 0 \le \left|\overbrace{\dfrac{\sin(\log x)}{\log x}}^{\to 0}\right| \le \left|\overbrace{\dfrac{1}{\log x}}^{\to 0}\right|$

(b) $x \to 1$ $\quad y = \log x \to 0 \quad \dfrac{\sin(\log x)}{\log x} = \dfrac{\sin y}{y} \to 1$

(c) $x \to e$ $\quad \dfrac{\sin(\log x)}{\log x} \to \dfrac{\sin(\log e)}{\log e} = \sin(1)$

(d) $x \to +\infty$ $\quad 0 \le \left|\overbrace{\dfrac{\sin(\log x)}{\log x}}^{\to 0}\right| \le \left|\overbrace{\dfrac{1}{\log x}}^{\to 0}\right|$

2) $\dfrac{\log(x^3+1)}{\log(x^2+1)}$

(a) $x \to -\infty$ $\quad \dfrac{\log\overbrace{(x^3+1)}^{\to -\infty}}{\log(x^2+1)}$ N.E.

(b) $x \to 0$ $\quad \dfrac{\log(x^3+1)}{\log(x^2+1)} = \dfrac{x^3 + o(x^3)}{x^2 + o(x^2)} = x\,\dfrac{1 + o(x^3)/x^3}{1 + o(x^2)/x^2} \to 0$

(c) $x \to 1$ $\quad \dfrac{\log(x^3+1)}{\log(x^2+1)} \to \dfrac{\log 2}{\log 2} = 1$

(d) $x \to +\infty$ $\quad \dfrac{\log(x^3+1)}{\log(x^2+1)} = \dfrac{\log x^3 + \log\left(1+\frac{1}{x^3}\right)}{\log x^2 + \log\left(1+\frac{1}{x^2}\right)} = \dfrac{3}{2}\,\dfrac{1 + \dfrac{\log\left(1+\frac{1}{x^3}\right)}{\log x}}{1 + \dfrac{\log\left(1+\frac{1}{x^2}\right)}{\log x}} \to \dfrac{3}{2}$

3) $\frac{\sin x}{x-\pi}$

(a) $x \to -\infty$ $\qquad 0 \le \left|\overset{\to 0}{\frac{\sin x}{x-\pi}}\right| \le \left|\overset{\to 0}{\frac{1}{x-\pi}}\right|$

(b) $x \to 0^+$ $\qquad \frac{\sin x}{x-\pi} \to \frac{\sin 0}{-\pi} = 0$

(c) $x \to \pi$ $\qquad y = \pi - x \to 0 \qquad \frac{\sin x}{x-\pi} = \frac{\sin(\pi - y)}{-y} = -\frac{\sin y}{y} \to -1$

(d) $x \to +\infty$ $\qquad 0 \le \left|\overset{\to 0}{\frac{\sin x}{x-\pi}}\right| \le \overset{\to 0}{\frac{1}{x-\pi}}$

4) $\frac{\cos x}{(2x-\pi)^2}$

(a) $x \to 0$ $\qquad \frac{\cos x}{(2x-\pi)^2} \to \frac{\cos 0}{(-\pi)^2} = \frac{1}{\pi^2}$

$x \to \frac{\pi}{2} \qquad y = \frac{\pi}{2} - x \to 0 \qquad \frac{\cos x}{(2x-\pi)^2} = \frac{\cos\left(\frac{\pi}{2} - y\right)}{(-2y)^2} = \frac{1}{4}\,\frac{\sin y}{y^2}$ N.E.

$x \to \frac{3}{2}\pi \qquad \frac{\cos x}{(2x-\pi)^2} \to \frac{\cos(3\pi/2)}{(2\pi)^2} = 0$

$x \to +\infty \qquad \overset{\to 0}{\frac{-1}{(2x-\pi)^2}} \le \overset{\to 0}{\frac{|\cos x|}{(2x-\pi)^2}} \le \overset{\to 0}{\frac{1}{(2x-\pi)^2}}$

5) $\frac{x^2+x-2}{\sqrt{x}-1}$

(a) $x \to 0^-$ $\qquad \frac{x^2+x-2}{\sqrt{x}-1} \to \frac{-2}{-1} = 2$

(b) $x \to 0^+$ $\qquad \frac{x^2+x-2}{\sqrt{x}-1} \to \frac{-2}{-1} = 2$

(c) $x \to 1$ $\qquad \frac{x^2+x-2}{\sqrt{x}-1} = \frac{(x+2)\cancel{(x-1)}}{\cancel{\sqrt{x}-1}}\,\frac{\sqrt{x}+1}{\cancel{\sqrt{x}+1}} \to 6$

(d) $x \to +\infty$ $\qquad \frac{x^2+x-2}{\sqrt{x}-1} = \overset{\to +\infty}{\frac{x^2}{\sqrt{x}}}\;\overset{\to 1}{\frac{1+1/x-2/x^2}{1-1/\sqrt{x}}} \to +\infty$

6) $\dfrac{\log(x^2-3)}{x^5-x-15}$

(a) $x\to-\infty$ $\quad \dfrac{\log(x^2-3)}{x^5-x-15} = \overset{\to 0}{\dfrac{\log(x^2-3)}{x^2-3}}\ \overset{\to 0}{\dfrac{x^2-3}{x^5-x-15}} \to 0$

(b) $x\to\sqrt{3}^{\,+}$ $\quad \dfrac{\log(x^2-3)\ \to -\infty}{x^5-x-15\ \to -\sqrt{3}-5} \longrightarrow +\infty$

(c) $x\to 2$ $\quad \dfrac{\log(x^2-3)}{x^5-x-15} = \dfrac{\log(1+(x^2-5))}{(x^2-5)}\ \dfrac{x^2-5}{x^5-x-15} =$

	1	0	0	-1	-15
2		2	5	8	15
	1	2	5	7	0

$x^5-x-15=(x-2)(x^3+2x^2+5x+7)$

$$= \overset{\to 1}{\dfrac{\log(1+(x^2-5))}{(x^2-5)}}\ \overset{\to 5/31}{\dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x^3+2x^2+5x+7)}} \to \frac{5}{31}$$

(d) $x\to+\infty$ $\quad \dfrac{\log(x^2-3)}{x^5-x-15} = \overset{\to 0}{\dfrac{\log(x^2-3)}{x^2-3}}\ \overset{\to 0}{\dfrac{x^2-3}{x^5-x-15}} \to 0$

7) $(x^2+3)^{1/\log x}$

(a) $x\to 0^+$ $\quad (\underset{\to 3}{x^2+3})^{\overset{\to 0}{1/\log x}} \to 1$

(b) $x\to 1^-$ $\quad (\underset{\to 5}{x^2+3})^{\overset{\to -\infty}{1/\log x}} \to 0$

(c) $x\to 1^+$ $\quad (\underset{\to 5}{x^2+3})^{\overset{\to +\infty}{1/\log x}} \to +\infty$

(d) $x\to+\infty$ $\quad (x^2+3)^{1/\log x} = e^{\frac{\log(x^2+3)}{\log x}\ \to 2} \longrightarrow e^2$

8) $(\sin x)^{\tan^2 x}$

(a) $x \to 0^+$ $\quad (\sin x)^{\tan^2 x} = e^{\overset{\to 0}{\tan^2 x \log \sin x}} \to 1$

$$\tan^2 x \log \sin x = \overset{\to 0}{\frac{\tan^2 x}{x}} \, \overset{\to 0}{\log (\sin x)^x} \to 0$$

$$(\sin x)^x = e^{x \log \sin x} = e^{x\left(\overset{\to 0}{\log \frac{\sin x}{x}} + \log x\right)} \to 1$$

(b) $x \to \pi/2^+$ $\quad x = \frac{\pi}{2} - y \quad y \to 0^-$

$$(\sin x)^{\tan^2 x} = (\cos y)^{1/\tan^2 y} = e^{\overset{\to -1/2}{\frac{\log \cos y}{\tan^2 y}}} \to \frac{1}{\sqrt{e}}$$

$$\frac{\log \cos y}{\tan^2 y} = \overset{\to 1}{\frac{y^2}{\tan^2 y}} \, \overset{\to -1/2}{\frac{\log \cos y}{y^2}} \to -1/2$$

$$\frac{\log \cos y}{y^2} = \frac{1}{2} \frac{\log \cos^2 y}{y^2} = \frac{1}{2} \frac{\log (1 - \sin^2 y)}{y^2} =$$

$$= \frac{1}{2} \overset{\to -1}{\frac{\log (1 - \sin^2 y)}{\sin^2 y}} \, \overset{\to 1}{\frac{\sin^2 y}{y^2}} \to -1/2$$

(c) $x \to \pi/2^-$ $\quad x = \frac{\pi}{2} - y \quad y \to 0^+$

$$(\sin x)^{\tan^2 x} = (\cos y)^{1/\tan^2 y} = e^{\overset{\to -1/2}{\frac{\log \cos y}{\tan^2 y}}} \to \frac{1}{\sqrt{e}}$$

(d) $x \to +\infty$

$$a_k = \frac{\pi}{6} + 2n\pi \to +\infty$$

$$(\sin a_k)^{\tan^2 a_k} \to \left(\frac{1}{2}\right)^{1/3} \qquad \rightsquigarrow \quad \text{N. E.}$$

$$b_n = \frac{\pi}{4} + 2n\pi \to +\infty$$

$$(\sin b_n)^{\tan^2 b_n} \to \frac{\sqrt{2}}{2}$$

9) $(\tan x)^{\cos x}$

(a) $x \to 0^+$ $\quad \underbrace{(\tan x)}_{\to 0}{}^{\overbrace{\cos x}^{\to 1}} \to 0$

(b) $x \to \pi/4$ $\quad \underbrace{(\tan x)}_{\to 1}{}^{\overbrace{\cos x}^{\to \sqrt{2}/2}} \to 1$

(c) $x \to \pi/2^-$ $\quad y = \frac{\pi}{2} - x \to 0^+$

$$(\tan x)^{\cos x} = \left(\frac{1}{\tan y}\right)^{\sin y} = e^{-\overbrace{\sin y \log \tan y}^{\to 0}} \to 1$$

$$\sin y \log \tan y = \frac{\sin y}{y} \cdot y\left(\log \frac{\tan y}{y} + \log y\right) \to 0$$

(d) $x \to -\pi^-$ $\quad y = \pi + x \to 0^-$

$$(\tan x)^{\cos x} = (\tan y)^{-\cos y} = \frac{1}{(\tan y)^{\cos y}} \to -\infty$$

10.a) $$\frac{(\log n)^n}{n^5 \cdot 5^n} = \frac{e^{n \log \log n}}{e^{5 \log n + n \log 5}} = e^{\overbrace{n \log\log n - 5 \log n - n \log 5}^{\to +\infty}} \to +\infty$$

$$n \log\log n - 5\log n - n \log 5 = n\Big(\underbrace{\log\log n}_{\to +\infty} - \underbrace{5\frac{\log n}{n}}_{\to 0} - \log 5\Big) \to +\infty$$

10.b) $$\frac{\log(n \log n)}{\log(n \log^3 n)} = \frac{\log(n \log n)}{\log(3 n \log n)} = \frac{\log(n \log n)}{\log(n \log n) + \log 3} \to 1$$

10.c) $$\frac{\sqrt{n + \sqrt{n^3+2}}}{\sqrt[5]{n^3 - 7}} = \frac{n^{3/4}}{n^{3/5}} \frac{\sqrt{n^{-1/2} + \sqrt{1 + 2/n^3}}}{\sqrt{1 - 7/n^3}} \to 1$$

11.a) $$\frac{27^{\sqrt[3]{n}}}{3^{\sqrt{n}}} = 3^{\overset{\to -\infty}{3\sqrt[3]{n}-\sqrt{n}}} \to 0$$

$$3\sqrt[3]{n}-\sqrt{n} = \overset{\to+\infty}{\sqrt{n}}\,(\overset{\to -1}{3n^{-1/6}-1}) \to -\infty$$

11.b) $$\frac{\log(1+3^n)}{\log(1+2^n)} = \frac{n\log 3+\log(1/3^n+1)}{n\log 2+\log(1/2^n+1)} \to \frac{\log 3}{\log 2}$$

11.c) $$(\log n)^n - n^{\log n} = n^{\log n}\left(\overset{\to+\infty}{\frac{(\log n)^n}{n^{\log n}}} - 1\right) \to +\infty$$

$$\frac{(\log n)^n}{n^{\log n}} = e^{\overset{\to+\infty}{n\log\log n-\log^2 n}} \to +\infty$$

$$n\log\log n-\log^2 n = n\left(\log\log n - \frac{\log^2 n}{n}\right) \to +\infty$$

12.a) $$\sqrt[n]{\sqrt[n]{n!}} = n!^{\frac{1}{n^2}} = e^{\overset{\to 0}{\frac{\log n!}{n^2}}} \to 1$$

$$\frac{\log n!}{n^2} \le \frac{\log n^n}{n^2} = \frac{\log n}{n} \to 0$$

12.b) $$\sqrt[n]{\sqrt[n]{n}-\sqrt[n]{2}} = e^{\overset{\to 0}{\frac{\log(\sqrt[n]{n}-\sqrt[n]{2})}{n}}} \to 1$$

$$\frac{\log(\sqrt[n]{n}-\sqrt[n]{2})}{n} = \frac{\log(\sqrt[n]{n})+\log(1-\sqrt[n]{2/n})}{n} =$$

$$= \overset{\to 0}{\frac{\log n}{n^2}} + \overset{\to 0}{\frac{\log(1-\sqrt[n]{2/n})}{n}} \to 0$$

12.c) $\sqrt[2n]{n!}\,\arcsin\dfrac{\sqrt{n+2}}{n}$

$$\sqrt[2n]{n!}\,\arcsin\frac{\sqrt{n+2}}{n}=\overset{\to 0}{\frac{\sqrt[2n]{n!}}{\sqrt{n}}}\ \overset{\to 1}{\frac{\sqrt{n+2}}{\sqrt{n}}}\ \overset{\to 1}{\frac{\arcsin\frac{\sqrt{n+2}}{n}}{\frac{\sqrt{n+2}}{n}}}\to 0$$

$$a_n=\frac{\sqrt[2n]{n!}}{\sqrt{n}}=\frac{\sqrt[2n]{n!}}{\sqrt[2n]{n^n}}=\left(\frac{n!}{n^n}\right)^{1/2n}\to 0 \qquad \sqrt[n]{a_n}=\left(\frac{n!}{n^n}\right)^{1/2}\to 0$$

13.a) $$\left(\frac{n}{n+6}\right)^n=\frac{1}{\left(1+\frac{6}{n}\right)^n}\to e^{-6}$$

13.b) $$\frac{n^{\sqrt{n}}}{(\log n)^{\log n}}=e^{\overset{\to+\infty}{(\sqrt{n}\log n-\log n\cdot\log\log n)}}\to+\infty$$

$$\sqrt{n}\log n-\log n\cdot\log\log n=\overset{\to+\infty}{\sqrt{n}\log n}\left(1-\overset{\to 0}{\frac{\log\log n}{\sqrt{n}}}\right)\to+\infty$$

$$\frac{\log\log n}{\sqrt{n}}=\overset{\to 0}{\frac{\log\log n}{\log n}}\ \overset{\to 0}{\frac{\log n}{\sqrt{n}}}\to 0$$

13.c) $$\frac{(n+3)^n+3^n}{(n+7)^n+10^n}=\frac{n^n}{n^n}\ \frac{(1+3/n)^n+(3/n)^n}{(1+7/n)^n+(10/n)^n}\to e^{-4}$$