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Limiti 2

Argomenti: limiti di successioni **Difficoltà**: ⋆⋆⋆

Prerequisiti: criteri del rapporto, della radice, del rapporto → radice

Calcolare i limiti delle seguenti successioni (esplicitando tutti i dettagli, almeno quando si svolge l'esercizio per la prima volta).

	A		B		C		D	
	Success.	Limite	Success.	Limite	Success.	Limite	Success.	Limite
1)	$\frac{3^n}{n^3}$	$+\infty$	$\frac{3^n}{n!}$	0	$\frac{n!}{n^n}$	0	$\frac{3^{n^2}}{n^n}$	$+\infty$
2)	$\frac{n!\cdot 2^n}{n^n}$	0	$\frac{n!\cdot 3^n}{n^n}$	$+\infty$	$\frac{n!\cdot 3^n}{n^{2n}}$	0	$\frac{n^{n^2}}{3^{n^3}}$	0
3)	$\frac{(n!)^2}{n^n}$	$+\infty$	$\frac{(2n)!}{n^n}$	$+\infty$	$\frac{(n!)^2}{(2n)!}$	0	$\frac{(2n)!}{3^{n^2}}$	0
4)	$\sqrt[n]{n}$	1	$\sqrt[n]{n!}$	$+\infty$	$\binom{3n}{n}$	$+\infty$	$\sqrt[n]{\binom{3n}{n}}$	$\frac{27}{4}$
5)	$\frac{\sqrt[n]{n!}}{n}$	$1/e$	$\frac{\sqrt[n]{(2n)!}}{n^2}$	$4/e^2$	$\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}$	$4/e$	$\frac{(2n)!}{n^{3n}}$	0

	A		B		C	
	Successione	Limite	Successione	Limite	Successione	Limite
6)	$2^n - n^2$	$+\infty$	$3^n - n!$	$-\infty$	$n^2 - n! + n^n$	$+\infty$
7)	$\frac{2^n + 5^n}{3^n + 4^n}$	$+\infty$	$\frac{n^3 + 2^n}{n^2 + 3^n}$	0	$\frac{2^n - n!}{n! + n^{22}}$	-1
8)	$\frac{33^n - n^{33}}{3^{n^2}}$	0	$\frac{1}{2^n}\left(1+\frac{1}{n}\right)^{n^2}$	$+\infty$	$\frac{1}{4^n}\left(1+\frac{1}{n}\right)^{n^2}$	0
9)	$\frac{n!(4n)!}{(2n)!(3n)!}$	$+\infty$	$\sqrt[n]{\frac{n!(4n)!}{(2n)!(3n)!}}$	$\frac{64}{27}$	$(5n)! - (2n)!(3n)!$	$+\infty$
10)	$\binom{3n}{n} - 6^n$	$+\infty$	$\binom{3n}{n} - 7^n$	$-\infty$	$\binom{3n}{n} - \binom{3n}{n+1}$	$-\infty$
11)	$\frac{n! + (3n)^n}{n! - (2n)^{2n}}$	0	$\frac{\sqrt{n!} + 2^{\sqrt{n}}}{3^n + n^3}$	$+\infty$	$\frac{5^n + (-2)^n}{4^n + (-3)^n}$	$+\infty$
12)	$\frac{n^{n!}}{(n!)^n}$	$+\infty$	$n^{n!} - (n!)^{n^3}$	$+\infty$	$\frac{(n!)^{2^n}}{(2^n)!}$	$+\infty$

1.A) $\dfrac{3^n}{n^3} \to +\infty$ $\qquad [3^n >> n^3]$

$$\frac{a_{n+1}}{a_n} = \frac{3^{\cancel{n}+1}}{(n+1)^3} \cdot \frac{n^3}{\cancel{3^n}} = \frac{3n^3}{(n+1)^3} \rightsquigarrow 3$$

1.B) $\dfrac{3^n}{n!} \to 0$ $\qquad [n! >> 3^n]$

$$\frac{a_{n+1}}{a_n} = \frac{3^{\cancel{n}+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{3^n}} = \frac{3}{n+1} \rightsquigarrow 0$$

1.C) $\dfrac{n!}{n^n} \to 0$ $\qquad [n^n >> n!]$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)!}}{(n+1)^{\cancel{n+1}}} \cdot \frac{n^n}{\cancel{n!}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \rightsquigarrow \frac{1}{e} < 1$$

1.D) $\dfrac{3^{n^2}}{n^n} \to +\infty$ $\qquad [3^{n^2} >> n^n]$

$$\sqrt[n]{a_n} = \frac{3^n}{n} \rightsquigarrow +\infty$$

2.A) $\dfrac{n! \cdot 2^n}{n^n} \to 0$

$$\sqrt[n]{a_n} = 2\sqrt[n]{\frac{n!}{n^n}} = 2\sqrt[n]{b_n} \qquad \frac{b_{n+1}}{b_n} \to \frac{1}{e} \text{ (vd. 1.C)}$$

$$\rightsquigarrow \sqrt[n]{a_n} = 2\sqrt[n]{b_n} \to 2/e < 1$$

2.B) $\dfrac{n! \cdot 3^n}{n^n} \to +\infty$

Vd. 2.A $\rightsquigarrow \sqrt[n]{a_n} = 3\sqrt[n]{b_n} \to 3/e > 1$

2.C) $\dfrac{n! \cdot 3^n}{n^{2n}} \to 0$

$$\sqrt[n]{a_n} = \underset{\to 0}{\frac{3}{n}} \overset{\to 1/e}{\sqrt[n]{\frac{n!}{n^n}}} \to 0 < 1$$

2.D) $\dfrac{n^{n^2}}{3^{n^3}} = \left(\dfrac{n^n}{3^{n^2}}\right)^n \to 0$

$$\sqrt[n]{a_n} = \frac{n^n}{3^{n^2}} \to 0 < 1$$

3.A) $\dfrac{(n!)^2}{n^n} \to +\infty$

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)\cancel{!}]^{\cancel{2}}}{(n+1)^{n+\cancel{1}}} \frac{n^n}{\cancel{(n!)^2}} = \overset{\to +\infty}{(n+1)} \underset{\to 1/e}{\left(\frac{n}{1+n}\right)^n} \to +\infty$$

3.B) $\dfrac{(2n)!}{n^n} \to +\infty$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)^{n+1}} \frac{n^n}{(2n)!} = \frac{\overset{2}{\cancel{(2n+2)}}(2n+1)\cancel{(2n)!}}{(n+1)^{n+\cancel{1}}} \frac{n^n}{\cancel{(2n)!}} =$$

$$= \overset{\to +\infty}{2(2n+1)} \overset{\to 1/e}{\left(\frac{n}{n+1}\right)^n} \to +\infty$$

3.C) $\dfrac{(n!)^2}{(2n)!} \to 0$

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)\cancel{!}]^2}{(2n+2)!} \frac{(2n)!}{\cancel{(n!)^2}} = \frac{(n+1)^2 \cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \to \frac{1}{4} < 1$$

3.D) $\dfrac{(2n)!}{3^{n^2}} \to 0$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{3^{(n+1)^2}} \frac{3^{n^2}}{(2n)!} = (2n+2)(2n+1) \frac{\cancel{3^{n^2}}}{\cancel{3^{n^2}} \cdot 3^{2n} \cdot 3} \to 0$$

5.A) $\sqrt[n]{n} \to 1$

$$\sqrt[n]{b_n} \quad b_n = n \qquad \frac{b_{n+1}}{b_n} = \frac{n+1}{n} \to 1 \leadsto \sqrt[n]{b_n} \to 1$$

5.B) $\sqrt[n]{n!} \to +\infty$

$\sqrt[n]{b_n}$ $\quad b_n = n!$ $\quad \frac{b_{n+1}}{b_n} = \frac{(n+1)!}{n!} = n+1 \to +\infty$

5.C) $\binom{3n}{n} \to +\infty$

$$\frac{a_{n+1}}{a_n} = \binom{3n+3}{n+1}\binom{3n}{n}^{-1} = \frac{(3n+3)!}{(n+1)\cancel{!}\,(2n+2)!}\,\frac{\cancel{n!}\,(2n)!}{(3n)!} =$$

$$= \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} \to \frac{27}{4} > 1$$

5.D) $\sqrt[n]{\binom{3n}{n}} \to \frac{27}{4}$

$\sqrt[n]{b_n}$ $\quad \frac{b_{n+1}}{b_n} \to \frac{27}{4}$ (Vd. 5.C)

5.A) $\frac{\sqrt[n]{n!}}{n} \to 1/e$

$a_n = \sqrt[n]{b_n}$ $\quad b_n = \frac{n!}{n^n}$ $\quad \frac{b_{n+1}}{b_n} \to \frac{1}{e}$ (Vd. 1.C)

5.B) $\frac{\sqrt[n]{(2n)!}}{n^2} \to \frac{4}{e^2}$

$a_n = \sqrt[n]{b_n}$ $\quad b_n = \frac{(2n)!}{n^{2n}}$ $\quad \frac{b_{n+1}}{b_n} = \frac{(2n+2)!}{(n+1)^{2n+2}}\,\frac{n^{2n}}{(2n)!} =$

$$= \underbrace{\frac{(2n+2)(2n+1)}{(n+1)^2}}_{\to 4} \cdot \underbrace{\left(\frac{n}{n+1}\right)^{2n}}_{\to 1/e^2} \to \frac{4}{e^2}$$

5.C) $\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} \to \frac{4}{e}$

$a_n = \sqrt[n]{b_n}$ $\quad b_n = \frac{(2n)!}{n^n\, n!}$ $\quad \frac{b_{n+1}}{b_n} = \frac{(2n+2)!}{(n+1)^{(n+1)}(n+1)\cancel{!}}\,\frac{n^n\,\cancel{n!}}{(2n)!} =$

$$= \underbrace{\frac{(2n+2)(2n+1)}{(n+1)^2}}_{\to 4} \underbrace{\left(\frac{n}{n+1}\right)^{n}}_{\to 1/e} \to \frac{4}{e}$$

5.D) $\dfrac{(2n)!}{n^{3n}} \to 0$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)^{(3n+3)}} \; \frac{n^{3n}}{(2n)!} = \overset{\to 0}{\frac{(2n+2)(2n+1)}{(n+1)^3}} \; \overset{\to 1/e^3}{\left(\frac{n}{n+1}\right)^{3n}} \to 0$$

6.A) $2^n - n^2 = \overset{\to +\infty}{2^n} \; \overset{\to 1}{\left(1 - \frac{n^2}{2^n}\right)} \to +\infty$

6.B) $3^n - n! = \overset{\to +\infty}{n!} \; \overset{\to -1}{\left(\frac{3^n}{n!} - 1\right)} \to -\infty$

6.C) $n^2 - n! + n^n = \overset{\to +\infty}{n^n} \; \overset{\to 1}{\left(\frac{n^2}{n^n} - \frac{n!}{n^n} + 1\right)} \to +\infty$

7.A) $\dfrac{2^n + 5^n}{3^n + 5^n} = \overset{\to +\infty}{\frac{5^n}{5^n}} \; \overset{\to 1}{\frac{(2/5)^n + 1}{(3/5)^n + 1}} \to +\infty$

7.B) $\dfrac{n^3 + 2^n}{n^2 + 3^n} = \overset{\to 0}{\frac{2^n}{3^n}} \; \overset{\to 1}{\frac{\frac{n^3}{2^n} + 1}{\frac{n^2}{3^n} + 1}} \to 0$

7.C) $\dfrac{2^n - n!}{n! + n^{22}} = \overset{1}{\frac{\cancel{n!}}{\cancel{n!}}} \; \overset{\to -1}{\frac{\frac{2^n}{n!} - 1}{1 + \frac{n^{22}}{n!}}} \to -1$

8.A) $\dfrac{33^n - n^{33}}{3^{n^2}} \quad \overset{\to 0}{\frac{33^n}{3^{n^2}}} \; \overset{\to 1}{\left(1 - \frac{n^{33}}{33^n}\right)} \to 0$

8.B) $\dfrac{1}{2^n}\left(1 + \dfrac{1}{n}\right)^{n^2} \to +\infty$

$$\sqrt[n]{a_n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n \to \frac{e}{2} > 1$$

8.C) $\dfrac{1}{5^n}\left(1 + \dfrac{1}{n}\right)^{n^2} \to 0 \qquad \sqrt[n]{a_n} \to \dfrac{e}{5} < 1$

9.A) $\frac{n!\,(5n)!}{(2n)!\,(3n)!} \to +\infty$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(5n+5)(5n+3)(5n+2)(5n+1)}{(2n+2)(2n+1)(3n+3)(3n+2)(3n+1)} \to \frac{5^5}{2^2\cdot 3^3} = \frac{256}{108} > 1$$

9.B) $\sqrt[n]{\frac{n!\,(5n)!}{(2n)!\,(3n)!}} \to \frac{65}{27}$ $\qquad a_n = \sqrt[n]{b_n} \qquad \frac{b_{n+1}}{b_n} \to \frac{65}{27}$

9.C) $$(5n)! - (2n)!\,(3n)! = \overset{\to+\infty}{(5n)!}\left(\overset{\to 1}{1 - \frac{(2n)!\,(3n)!}{(5n)!}}\right) \to +\infty$$

$$b_n = \frac{(2n)!\,(3n)!}{(5n)!} \to 0$$

$$\frac{b_{n+1}}{b_n} = \frac{(2n+2)(2n+1)(3n+3)(3n+2)(3n+1)}{(5n+5)(5n+5)(5n+3)(5n+2)(5n+1)} = \frac{105}{3125} < 1$$

10.A) $$\binom{3n}{n} - 6^n = \overset{\to+\infty}{6^n}\left(\overset{\to+\infty}{\frac{1}{6^n}\binom{3n}{n}} - 1\right) \to +\infty$$

$$b_n = \frac{1}{6^n}\binom{3n}{n} \qquad \frac{b_{n+1}}{b_n} = \frac{1}{6}\,\frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} \to \frac{27}{25} > 1$$

10.B) $$\binom{3n}{n} - 7^n = \overset{\to+\infty}{7^n}\left(\overset{\to -1}{\frac{1}{7^n}\binom{3n}{n} - 1}\right) \to -\infty$$

$$b_n = \frac{1}{7^n}\binom{3n}{n} \qquad \frac{b_{n+1}}{b_n} = \frac{1}{7}\,\frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} \to \frac{27}{28} < 1$$

10.C) $$\binom{3n}{n} - \binom{3n}{n+1} = -\binom{3n+1}{n+1} \to -\infty$$

$$\binom{3n}{n+1} = \binom{3n+1}{n+1} - \binom{3n}{n} \to a_n = -\binom{3n+1}{n+1}$$

$$b_n = \binom{3n+1}{n+1} \to +\infty$$

$$\frac{b_{n+1}}{b_n} = \frac{(3n+5)!}{(n+2)!(2n+2)!} \cdot \frac{(n+2)!(2n)!}{(3n+2)!} = \frac{(3n+5)(3n+3)(3n+2)}{(n+2)(2n+2)(2n+1)} \to \frac{27}{5}$$

11.A) $$\frac{n! + (3n)^n}{n! - (2n)^{2n}} = \overbrace{\frac{(3n)^n}{(2n)^{2n}}}^{\to 0} \; \frac{\overbrace{\frac{n!}{(3n)^n} + 1}^{\to -1}}{\frac{n!}{(2n)^{2n}} - 1} \to 0$$

11.B) $$\frac{\sqrt{n!} + 2^{\sqrt{n}}}{3^n + n^3} = \overbrace{\frac{\sqrt{n!}}{3^n}}^{\to +\infty} \; \frac{1 + \overbrace{\frac{2^{\sqrt{n}}}{\sqrt{n!}}}^{\to 1}}{1 + \frac{n^3}{3^n}} \to +\infty$$

$$b_n = \frac{\sqrt{n!}}{2^{\sqrt{n}}} = \sqrt{\frac{n!}{2^{2\sqrt{n}}}} \to +\infty$$

$$c_n = \frac{\sqrt{n!}}{3^n} = \sqrt{\frac{n!}{3^{2n}}} \to +\infty$$

11.C) $$\frac{5^n + (-2)^n}{5^n + (-3)^n} = \overbrace{\frac{5^n}{5^n}}^{\to +\infty} \; \frac{\overbrace{1 + \left(\frac{-2}{5}\right)^n}^{\to 1}}{1 + \left(\frac{-3}{5}\right)^n} \to +\infty$$

12.A) $$\frac{n^{n!}}{n!^n} \to +\infty$$

$$\frac{n^{n!}}{n!^n} \geq \frac{n^{n!}}{(n^n)^n} = \frac{n^{n!}}{n^{n^2}} = n^{\overbrace{(n! - n^2)}^{\to +\infty}} \to +\infty$$

12.B) $$n^{n!} - (n!)^{n^3} = \overbrace{n^{n!}}^{\to +\infty} \Bigg(1 - \overbrace{\frac{(n!)^{n^3}}{n^{n!}}}^{\to 1} \Bigg) \to +\infty$$

$$0 \leq \overbrace{\frac{(n!)^{n^3}}{n^{n!}}}^{\to 0} \leq \frac{(n^n)^{n^3}}{n^{n!}} = \overbrace{\frac{n^{n^5}}{n^{n!}}}^{\to 0}$$

12.c) $\dfrac{(n!)^{2^n}}{(2^n)!} \longrightarrow +\infty$

$$\frac{(n!)^{2^n}}{(2^n)!} \geq \frac{(n!)^{2^n}}{(2^n)^{2^n}} = \left(\frac{n!}{2^n}\right)^{2^n} \longrightarrow +\infty$$

$\rightarrow +\infty$