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Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Scritto d'esame di Analisi Matematica II

Pisa, ?? ?? ????

1. Siano $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z = 1\}$ e $f(x, y, z) = x^4 + y^2 - z$.
   (a) Mostrare che D non è limitato.
   (b) Calcolare estremo inferiore e superiore di f in D precisando se si tratta di minimo e/o massimo e calcolando anche gli eventuali punti di massimo/minimo.

2. Sia V il solido definito da

$$V := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y \leq 1,\ y \geq 0,\ 0 \leq z \leq 1\}.$$

   Calcolare

$$\int_V (x - z)\, dx\, dy\, dz, \qquad \int_V |x - z|\, dx\, dy\, dz.$$

3. Sia $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1,\ x \geq 0,\ y \geq 0\}$.
   (a) Provare che

$$\int_B \frac{\arctan(xy)}{(x + y)^4} dx\, dy < +\infty.$$

   (b) Stabilire per quali $\alpha > 0$ si ha

$$\int_B \frac{\arctan(xy)}{(x + y)^\alpha} dx\, dy < +\infty.$$

4. Sia γ la curva parametrizzata da $\gamma(t) = (t^2(\pi - t), \sin t)$ con $0 \leq t \leq \pi$.
   (a) Provare che γ è chiusa e semplice e farne un disegno approssimativo.
   (b) Calcolare l'area del dominio racchiuso da γ.

Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato.
Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. Siano $D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z=1\}$ e $f(x,y,z) = x^4+y^2-z$.

(a) Mostrare che D non è limitato.

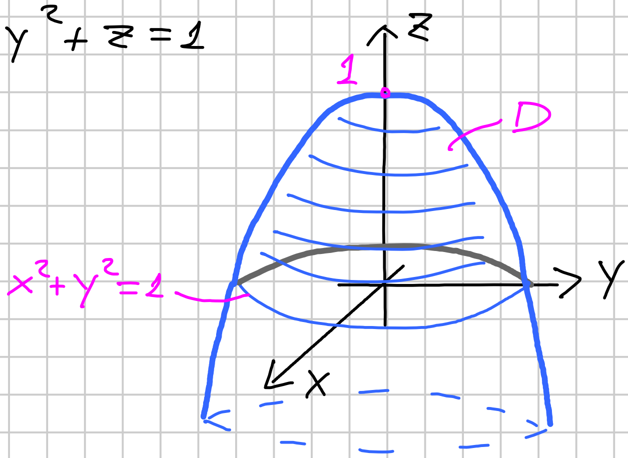
(b) Calcolare estremo inferiore e superiore di f in D precisando se si tratta di minimo e/o massimo e calcolando anche gli eventuali punti di massimo/minimo.

(a) $x=0 \rightsquigarrow y^2 = 1-z$ $\qquad D: x^2+y^2+z=1$

$y = \pm\sqrt{1-z}$

$z \to -\infty \rightsquigarrow y \rightsquigarrow \pm\infty$

$\rightsquigarrow$ D NON È LIMITATO



(b) MODO 1 – SOSTITUZIONE DEL VINCOLO

$$\begin{cases} D: x^2+y^2+z=1 \qquad z = 1-x^2-y^2 \\ f(x,y,z) = x^4+y^2-z = x^4+y^2-1+x^2+y^2 = x^4+x^2+2y^2-1 \end{cases}$$

$\rightsquigarrow$ STUDIAMO $g(x,y) = x^4+x^2+2y^2-1 \qquad (x,y)\in\mathbb{R}^2$

$$\lim_{x^2+y^2\to+\infty} g(x,y) \overset{y^2=v^4}{=} \lim_{x^2+v^2\to+\infty} x^4+x^2+2v^4-1 =$$

$$= \lim_{\rho\to+\infty} \underbrace{\rho^4}_{\rightsquigarrow+\infty}\Big(\underbrace{\cos^4\theta+2\sin^4\theta}_{\geq c>0} + \underbrace{\frac{\cos^2\theta}{\rho^2}}_{\rightsquigarrow 0} - \underbrace{\frac{1}{\rho^4}}_{\rightsquigarrow 0}\Big) = +\infty \quad \begin{cases} \exists \text{ MINIMO} \\ \text{SUP}(g) = +\infty \end{cases}$$

$$g(x,y) = \underbrace{x^4}_{\geq 0}+\underbrace{x^2}_{\geq 0}+\underbrace{2y^2}_{\geq 0}-1 \rightsquigarrow g(0,0) = -1 \equiv \text{MIN}(g)/\text{INF}(g)$$

$$\rightsquigarrow \begin{cases} \text{SUP}(f_D) = +\infty \\ \text{INF}(f_D) = \text{MIN}(f_D) = -1 \quad \text{IN} \quad P=(0,0,-1) \end{cases}$$

MODO 2 – MOLTIPLICATORI DI LAGRANGE (PER DET. MINIMO)

$\Phi(x,y,z) = x^2+y^2+z-1=0$

SISTEMA 1 $\begin{cases} \Phi_x = 2x = 0 \\ \Phi_y = 2y = 0 \\ \Phi_z = 1 = 0 \\ \Phi = 0 \end{cases} \rightsquigarrow \emptyset$

SISTEMA 2
$$\begin{cases} f_x = 5x^3 = \lambda \cdot 2x \rightsquigarrow 5x^3+2x=0 \quad x(\underbrace{5x^2+2}_{>0})=0 \quad x=0 \\ f_y = 2y = \lambda 2y \rightsquigarrow 2y=-2y \rightsquigarrow y=0 \\ f_z = -1 = \lambda \cdot 1 \rightsquigarrow \lambda=-1 \\ \Phi = x^2+y^2+z-1=0 \rightsquigarrow z=1 \end{cases}$$

$\rightsquigarrow$ $INF(f_D) = MIN(f_D) = -1$ IN $P=(0,0,-1)$

2. Sia V il solido definito da

$$V := \{(x,y,z) \in \mathbb{R}^3 : x^2+y \le 1,\ y \ge 0,\ 0 \le z \le 1\}.$$

Calcolare

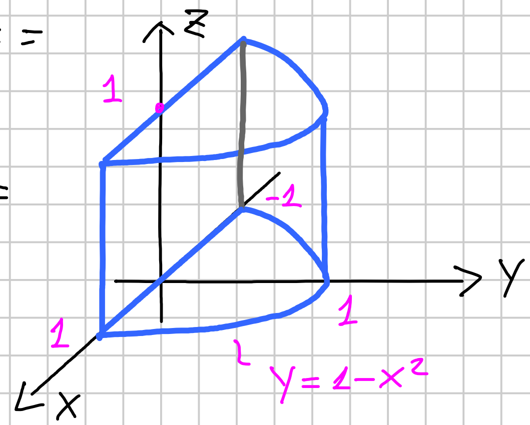
$$\int_V (x-z)\,dx\,dy\,dz, \qquad \int_V |x-z|\,dx\,dy\,dz.$$

$$\int_V (x-z)\,dx\,dy\,dz = \underbrace{\cancel{\int_V x\,dx\,dy\,dz}}_{=0} - \int_V z\,dx\,dy\,dz =$$

$$= -\int_V z\,dx\,dy\,dz = -\int_0^1\int_{-1}^1\int_0^{1-x^2} z\,dy\,dx\,dz =$$

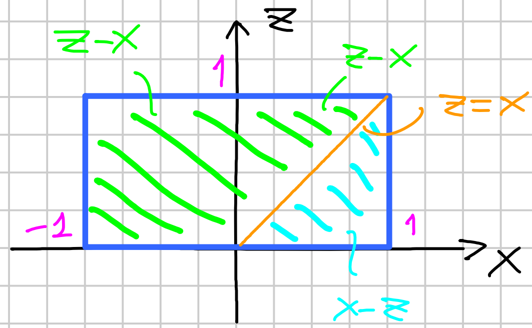
$$= -\int_0^1 z\,dz \int_{-1}^1 (1-x^2)\,dx = -\frac{1}{2}\left[x-\frac{x^3}{3}\right]_{-1}^1 =$$

$$= -\frac{1}{2}\left(1-\frac{1}{3}+1-\frac{1}{3}\right) = -\frac{2}{3}$$



VERIFICA: $-\int_V z\,dx\,dy\,dz = z_G \cdot V = -\frac{1}{2}\cdot\left(\frac{4}{3}\cdot 1\right) = -\frac{2}{3}$

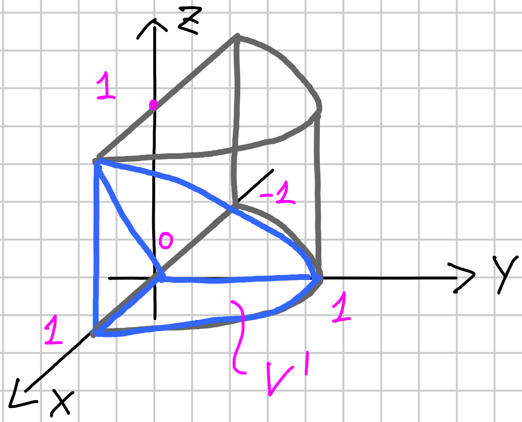
$$\int_V |x-z|\,dx\,dy\,dz$$



$$|x-z| = \begin{cases} x-z & x-z \ge 0 \quad z \le x \\ z-x & x-z \le 0 \quad z \ge x \end{cases}$$

$$\int_V |x-z|\,dx\,dy\,dz = \int_V (z-x)\,dx\,dy\,dz + 2\int_{V'} (x-z)\,dx\,dy\,dz$$

$$\int_V (z-x)\,dx\,dy\,dz = -\int_V (x-z)\,dx\,dy\,dz = \frac{2}{3}$$



$$\int_{V'} (x-z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_0^{1-x^2} (x-z)\,dy\,dz\,dx =$$

$$= \int_0^1 \int_0^x (x-z)(1-x^2)\,dz\,dx = \int_0^1 (1-x^2)\left[xz - \frac{z^2}{2}\right]_0^x dx =$$

$$= \int_0^1 (1-x^2)\left(x^2 - \frac{x^2}{2}\right)dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 =$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{2}\,\frac{2}{15} = \frac{1}{15}$$

$$\int_V |x-z|\,dx\,dy\,dz = \frac{2}{3} + 2\cdot\frac{1}{15} = \frac{12}{15} = \frac{4}{5}$$

3. Sia $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1, \, x \geq 0, \, y \geq 0\}$.

(a) Provare che

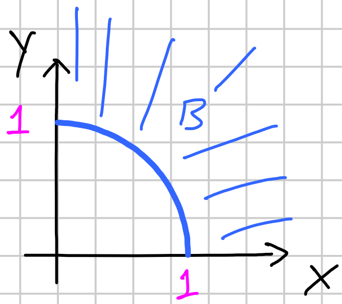
$$\int_B \frac{\arctan(xy)}{(x+y)^4} dx\, dy < +\infty.$$

(b) Stabilire per quali $\alpha > 0$ si ha

$$\int_B \frac{\arctan(xy)}{(x+y)^\alpha} dx\, dy < +\infty.$$

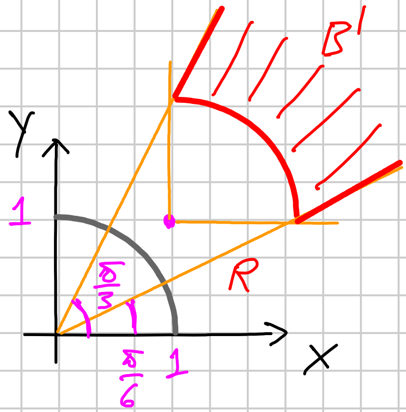
(a)

$$\int_B \frac{\arctan(xy)}{(x+y)^4} \, dx\, dy$$



$$\leq \int_B \frac{\pi/2}{(x+y)^4} \, dx\, dy = \int_0^{\pi/2} \int_1^{+\infty} \frac{\pi/2}{\rho^{\cancel{4}\,3} (\cos\theta + \sin\theta)^4} \cancel{\rho}\, d\rho\, d\theta =$$

$$= \int_0^{\pi/2} \underbrace{\frac{\pi/2}{(\cos\theta+\sin\theta)^4}}_{> c > 0} d\theta \int_1^{+\infty} \frac{1}{\rho^3} d\rho = M \int_1^{+\infty} \frac{1}{\rho^3} d\rho < +\infty$$



(b)

$$\int_B \frac{\arctan(xy)}{(x+y)^\alpha} \, dx\, dy$$

$$\leq \int_B \frac{\pi/2}{(x+y)^\alpha} \, dx\, dy = \ldots = M \int_1^{+\infty} \frac{1}{\rho^{\alpha-1}} d\rho < +\infty \text{ PER } \alpha - 1 > 1$$

PER $\alpha > 2$ CONVERGE

$$\geq \int_{B'} \frac{\pi/4}{(x+y)^\alpha} \, dx\, dy = \int_{\pi/6}^{\pi/3} \int_R^{+\infty} \frac{\pi/4}{\rho^\alpha (\cos\theta + \sin\theta)^\alpha} \rho\, d\rho\, d\theta =$$

$$= \int_{\pi/6}^{\pi/3} \frac{\pi/4}{(\cos\theta+\sin\theta)^\alpha} d\theta \int_R^{+\infty} \frac{1}{\rho^{\alpha-1}} d\rho = +\infty \quad \alpha - 1 \leq 1 \rightsquigarrow$$

PER $\alpha \leq 2$ DIVERGE

$\rightsquigarrow$ CONVERGE $\iff \alpha > 2$

4. Sia γ la curva parametrizzata da $\gamma(t) = (t^2(\pi - t), \sin t)$ con $0 \le t \le \pi$.

(a) Provare che γ è chiusa e semplice e farne un disegno approssimativo.

(b) Calcolare l'area del dominio racchiuso da γ.

(a) $\gamma(0) = (0,0) = \gamma(\pi) = (0,0) \rightsquigarrow$ γ È CHIUSA

$s \neq t$ $\begin{cases} t^2(\pi - t) = s^2(\pi - s) \\ \operatorname{sen} t = \operatorname{sen} s \end{cases}$ $\begin{cases} t^2\pi - t^3 = s^2\pi - s^3 \\ \operatorname{sen} t = \operatorname{sen} s \end{cases}$ $\begin{cases} \pi(t^2 - s^2) = t^3 - s^3 \\ \operatorname{sen} t = \operatorname{sen} s \end{cases}$

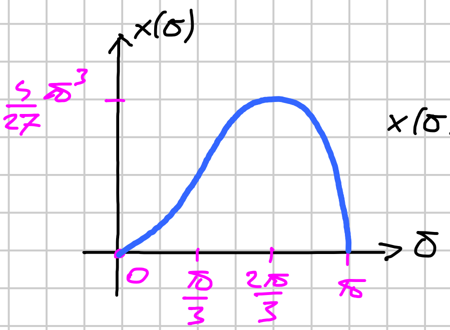
$\begin{cases} \pi(\cancel{t-s})(t+s) = (\cancel{t-s})(t^2 + ts + s^2) \\ \operatorname{sen} t = \operatorname{sen} s \rightsquigarrow t = \pi - s \end{cases}$

$\rightsquigarrow \pi(\pi - s + s) = (\pi - s)^2 + (\pi - s)s + s^2$

$\cancel{\pi^2} = \cancel{\pi^2} - 2s\pi + s^2 + s\pi - \cancel{s^2} + \cancel{s^2}$

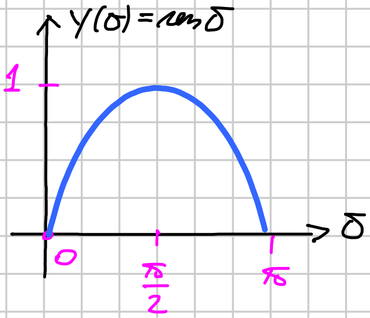
$s^2 - s\pi = 0 \quad s(s - \pi) = 0$

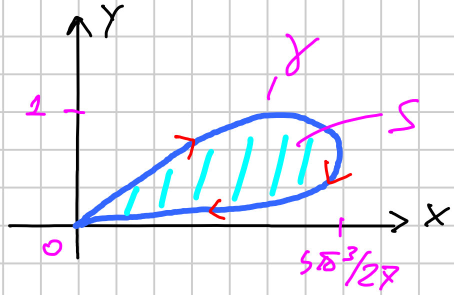
$\rightsquigarrow \begin{cases} s = 0 \rightsquigarrow t = \pi \\ s = \pi \rightsquigarrow t = 0 \end{cases} \rightsquigarrow$ γ È SEMPLICE



$x(t) = t^2(\pi - t)$ $\begin{cases} x'(t) = 2\pi t - 3t^2 = t(2\pi - 3t) = 0 \begin{cases} t = 0 \\ t = 2\pi/3 \end{cases} \\ x''(t) = 2\pi - 6t = 0 \quad t = \pi/3 \end{cases}$

$x_{MAX} = x\left(\frac{2\pi}{3}\right) = \frac{4\pi^2}{9} \cdot \left(\pi - \frac{2\pi}{3}\right) = \frac{5}{27}\pi^3$





(b) GAUSS-GREEN: $\vec{E}(0,y) \rightsquigarrow AREA = -\int_{\partial S} y\,dx$

$$x(\sigma) = \sigma^2(\pi - \sigma) \qquad y(\sigma) = \operatorname{sen}\sigma \qquad dx = (2\pi\sigma - 3\sigma^2)\,d\sigma$$

<u>OSS</u> SEGNO DI PERCORRENZA ORARIO $\rightsquigarrow$ AGGIUNGO SEGNO "$-$"

$$\int_S y\,dx = \int_0^\pi (2\pi\sigma - 3\sigma^2)\operatorname{sen}\sigma\,d\sigma = 2\pi\int_0^\pi \sigma\operatorname{sen}\sigma\,d\sigma - 3\int_0^\pi \sigma^2\operatorname{sen}\sigma\,d\sigma$$

$$\int_0^\pi \overset{f}{\sigma}\,\overset{g'}{\operatorname{sen}\sigma}\,d\sigma = \Big[-\sigma\cos\sigma\Big]_0^\pi + \int_0^\pi \cos\sigma\,d\sigma = -\pi(-1) + \Big[\operatorname{sen}\sigma\Big]_0^\pi = \pi$$

$$\int_0^\pi \overset{f}{\sigma^2}\,\overset{g'}{\operatorname{sen}\sigma}\,d\sigma = \Big[-\sigma^2\cos\sigma\Big]_0^\pi + \int_0^\pi \overset{l}{2\sigma}\,\overset{m'}{\cos\sigma}\,d\sigma =$$

$$= -\pi^2(-1) + \Big[2\sigma\operatorname{sen}\sigma\Big]_0^\pi - \int_0^\pi 2\operatorname{sen}\sigma\,d\sigma =$$

$$= \pi^2 - 2\Big[-\cos\sigma\Big]_0^\pi = \pi^2 - 4$$

$$\rightsquigarrow AREA = 2\pi\cdot\pi - 3(\pi^2 - 4) = 12 - \pi^2$$