Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Scritto d'esame di Analisi Matematica II

Pisa, ?? ?? ????

1. (a) Calcolare
$$\lim_{x^2+y^2\to+\infty} \frac{x^2y}{1+x^4+y^4}.$$
   (b) Stabilire per quali $\alpha > 0$ esiste
$$\lim_{x^2+y^2\to+\infty} \frac{x^2y}{1+|x|^\alpha+y^4}$$
   ed in caso affermativo calcolarlo.

2. Siano $D = \{(x,y) \in \mathbb{R}^2 : x^3 + y^4 \leq 2,\ 0 \leq y \leq x\}$ e $f(x,y) = 3x^2 + 4y^2$. Calcolare estremo inferiore e superiore di f in D precisando se si tratta di minimo e/o massimo e calcolando anche gli eventuali punti di massimo/minimo.

3. Sia D il dominio di $\mathbb{R}^2$ definito da
$$D := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2x,\ x \geq \frac{1}{2}\right\}.$$
   Calcolare area(D).

4. Sia S la superficie di $\mathbb{R}^3$ definita da
$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z = 1,\ z \geq 0\}$$
   orientata prendendo in $(0,0,1)$ la normale con direzione $(0,0,1)$. Sia $F(x,y,z) = (x+y, e^{z^{2^2}}, -z+1)$. Calcolare il flusso di F attraverso S (cioè $\int_S (F,\nu)\,d\sigma$).

Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato.
Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. (a) Calcolare

$$\lim_{x^2+y^2\to+\infty} \frac{x^2y}{1+x^4+y^4}.$$

(b) Stabilire per quali $\alpha > 0$ esiste

$$\lim_{x^2+y^2\to+\infty} \frac{x^2y}{1+|x|^\alpha+y^4}$$

ed in caso affermativo calcolarlo.

(a)

$$\lim_{x^2+y^2\to+\infty} \frac{x^2y}{1+x^4+y^4} = \lim_{\rho\to+\infty} \frac{\rho^3\cos^2\theta\sin\theta}{1+\rho^4(\cos^4\theta+\sin^4\theta)} =$$

$$= \lim_{\rho\to+\infty} \frac{\cancel{\rho^4}}{\cancel{\rho^4}}\, \frac{\overbrace{\dfrac{1-\sin^3\theta}{\rho}}^{\rightsquigarrow 0}}{\underbrace{\dfrac{1}{\rho^4}}_{\rightsquigarrow 0}+\underbrace{(\cos^4\theta+\sin^4\theta)}_{>c>0}} = 0$$

(b) PONIAMO: $\mu^4 = |x|^\alpha \qquad x^2 = |x|^2 = \mu^{\frac{8}{\alpha}} \qquad v = y$

$$\lim_{x^2+y^2\to+\infty} \frac{x^2y}{1+|x|^\alpha+y^4} = \lim_{\mu^2+v^2\to+\infty} \frac{\mu^{\frac{8}{\alpha}}v}{1+\mu^4+v^4} =$$

$a = \dfrac{8}{\alpha} \qquad a > 0$

$$= \lim_{\rho\to+\infty} \frac{\rho^{a+1}(\cos\theta)^a\sin\theta}{1+\rho^4(\cos^4\theta+\sin^4\theta)} =$$

$a < 3 \rightsquigarrow \alpha > \dfrac{8}{3}$:

$$= \lim_{\rho\to+\infty} \frac{\cancel{\rho^4}}{\cancel{\rho^4}}\, \frac{\overbrace{\rho^{(a-3)}}^{\rightsquigarrow 0}(\cos\theta)^{8/\alpha}\sin\theta}{\underbrace{1/\rho^4}_{\rightsquigarrow 0}+\underbrace{(\cos^4\theta+\sin^4\theta)}_{>c>0}} = 0$$

$a = 3 \rightsquigarrow \alpha = \dfrac{8}{3}$:

$$= \lim_{\rho\to+\infty} \frac{\cancel{\rho^4}}{\cancel{\rho^4}}\, \frac{(\cos\theta)^{8/\alpha}\sin\theta}{\underbrace{1/\rho^4}_{\rightsquigarrow 0}+\underbrace{(\cos^4\theta+\sin^4\theta)}_{>c>0}}$$

NON ESISTE

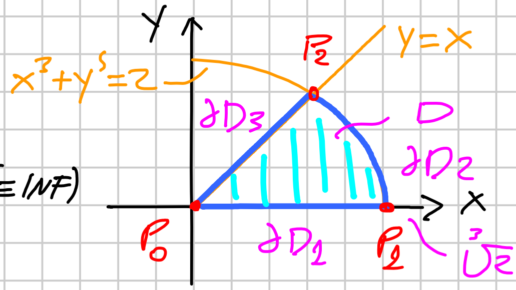
$a > 3 \rightsquigarrow \alpha < \dfrac{8}{3}$:

$$\left.\begin{aligned} &= \lim_{\substack{\mu^2+v^2\to+\infty \\ v=0}} \frac{\mu^a v}{1+\mu^4+v^4} = 0 \\ &= \lim_{\substack{\mu^2+v^2\to+\infty \\ \mu=v}} \frac{\mu^{a+1}}{1+2\mu^4} = +\infty \end{aligned}\right\}$$

NON ESISTE

2. Siano $D = \{(x,y) \in \mathbb{R}^2 : x^3 + y^4 \le 2,\ 0 \le y \le x\}$ e $f(x,y) = 3x^2 + 4y^2$. Calcolare estremo inferiore e superiore di f in D precisando se si tratta di minimo e/o massimo e calcolando anche gli eventuali punti di massimo/minimo.

$f(x,y) = 3x^2 + 4y^2$



D È COMPATTO $\rightsquigarrow \exists$ MAX($\equiv$SUP) MIN($\equiv$INF)

PUNTI STAZIONARI INTERNI:

$$\begin{cases} f_x = 6x = 0 \\ f_y = 8y = 0 \end{cases} \rightsquigarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \rightsquigarrow P_0 \in \partial D \qquad \nexists \text{ P.TI STAZIONARI INTERNI}$$

STUDIO DEL BORDO: $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$

$$\partial D_1 : y = 0 \rightsquigarrow g_1(x) = f(x,0) = 3x^2 \quad \begin{cases} \text{MIN IN } P_0 \\ \text{MAX IN } P_1 \end{cases}$$

$$\partial D_3 : y = x \rightsquigarrow g_3(x) = f(x,x) = 7x^2 \quad \begin{cases} \text{MIN IN } P_0 \\ \text{MAX IN } P_2 \end{cases}$$

∂D_2: MOLTIPLICATORI DI LAGRANGE

$$\Phi(x,y) = x^3 + y^4 - 2 = 0$$

SIST. 1
$$\begin{cases} \Phi_x = 3x^2 = 0 \\ \Phi_y = 4y^3 = 0 \\ \Phi = x^3 + y^4 - 2 = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = 0 \\ -2 = 0 \end{cases} \rightsquigarrow \emptyset$$

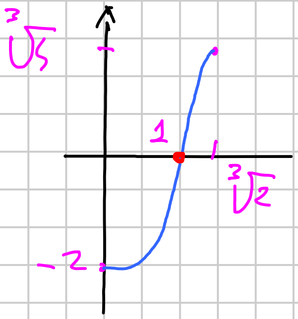
SIST. 2
$$\begin{cases} f_x = \cancel{6}^{\,2} x = \lambda \cdot \cancel{3} x^2 \\ f_y = \cancel{8}^{\,2} y = \lambda \cdot \cancel{4} y^3 \\ \Phi = x^3 + y^4 - 2 = 0 \end{cases} \quad \lambda x^2 \cdot y = \lambda y^3 \cdot x \qquad \lambda \cdot xy(x - y^2) = 0$$

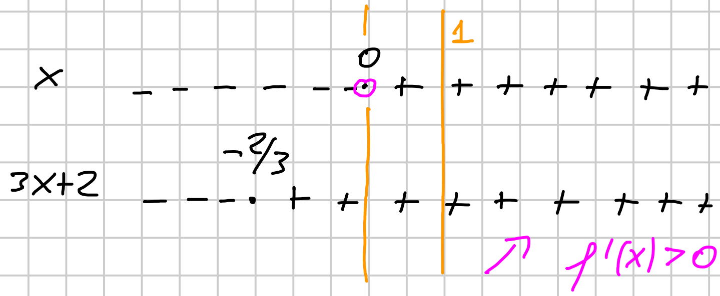
$$\rightsquigarrow \begin{cases} \lambda = 0 \\ xy = 0 \\ x - y^2 = 0 \end{cases}$$

$$\begin{cases} \lambda = 0 \rightsquigarrow \emptyset \\ xy = 0 \quad \begin{cases} x = 0 \rightsquigarrow y = \pm\sqrt[5]{2} \notin \partial D \\ y = 0 \rightsquigarrow x = \sqrt[3]{2} \rightsquigarrow P_1 \end{cases} \\ x - y^2 = 0 \rightsquigarrow x^3 + x^2 - 2 = 0 \end{cases}$$

$s(x) = x^3 + x^2 - 2 \qquad x \in [1, \sqrt[3]{2}] \qquad s(1) = 0 \qquad s(\sqrt[3]{2}) = \sqrt[3]{5} > 0$

$s'(x) = 3x^2 + 2x = x(3x+2)$





$\rightsquigarrow s(1) = 0 \qquad \rightsquigarrow x = 1 \quad y = 1 \quad \rightsquigarrow P_2 = (1,1)$

$f(P_0) = 0 \qquad f(P_1) = 3\sqrt[3]{5} \qquad f(P_2) = 7 > 3\sqrt[3]{5}$

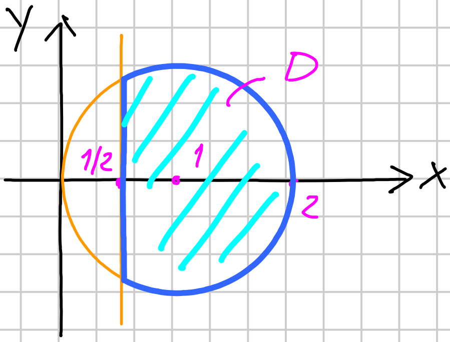
$$\rightsquigarrow \begin{cases} \max f_D = \sup f_D = 7 \quad \text{IN } P_2 \\ \min f_D = \inf f_D = 0 \quad \text{IN } P_0 \end{cases}$$

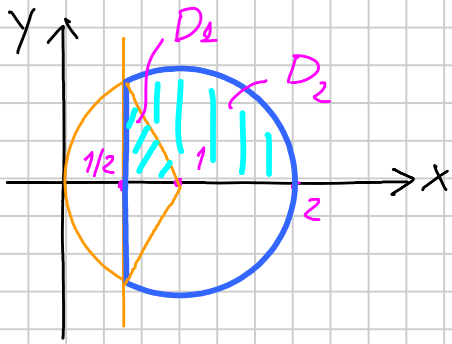
3. Sia D il dominio di $\mathbb{R}^2$ definito da

$$D := \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2x,\ x \geq \frac{1}{2} \right\}.$$

Calcolare area(D).

$$x^2+y^2-2x=0 \qquad \begin{cases} (x-x_0)^2+(y-y_0)^2=R^2 \\ x^2+y^2-2x_0x-2y_0y=R^2-x_0^2-y_0^2 \end{cases} \rightsquigarrow \begin{cases} x_0=1 \quad y_0=0 \\ R=1 \end{cases}$$





$$\text{AREA}(D) = 2\,\text{AREA}(D_1) + 2\,\text{AREA}(D_2)$$

$$\begin{cases} \text{AREA}(D_1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} \\ \text{AREA}(D_2) = \frac{1}{3}\pi \cdot 1^2 = \frac{\pi}{3} \end{cases}$$

$$\rightsquigarrow \text{AREA}(D) = \frac{\sqrt{3}}{4} + \frac{2}{3}\pi$$

4. Sia S la superficie di $\mathbb{R}^3$ definita da

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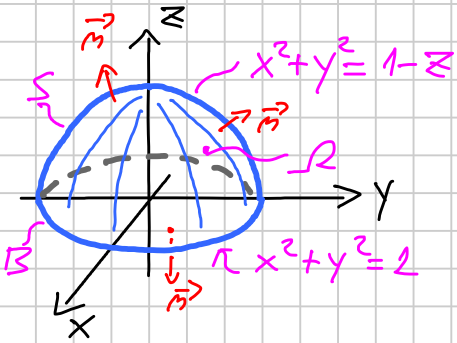
orientata prendendo in $(0,0,1)$ la normale con direzione $(0,0,1)$. Sia $F(x,y,z) = (x+y, e^{z^2}, -z+1)$. Calcolare il flusso di F attraverso S (cioè $\int_S (F,\nu)\,d\sigma$).

$S:\ x^2+y^2+z=1\ ,\ z \geq 0$



NORMALE IN $P=(1,0,0)$: $V_P=(2,0,1)$

TEOREMA DI GAUSS-GREEN:

$$\int_\Omega \operatorname{div}\vec{F}\,dx\,dy\,dz = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\,d\sigma = \int_S \vec{F}\cdot\vec{n}\,d\sigma + \int_B \vec{F}\cdot\vec{n}\,d\sigma$$

$$\rightsquigarrow \int_S \vec{F}\cdot\vec{n}\,d\sigma = \int_\Omega \operatorname{div}\vec{F}\,dx\,dy\,dz - \int_B \vec{F}\cdot\vec{n}\,d\sigma$$

$$\operatorname{div}\vec{F} = 1+0-1=0 \quad \rightsquigarrow \int_\Omega \operatorname{div}\vec{F}\,dx\,dy\,dz = 0$$

$$\int_B \vec{F}\cdot\vec{n}\,d\sigma = \int_B \langle (x+y,1,1),(0,0,-1)\rangle\,d\sigma = -\int_B d\sigma = -\pi$$

$$\rightsquigarrow \int_S \vec{F}\cdot\vec{n}\,d\sigma = \pi$$