Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Scritto d'esame di Analisi Matematica II

Pisa, ?? ?? ????

1. Sia $f(x, y, z) = x + y + z$ e sia D il dominio di $\mathbb{R}^3$ definito da
$$D = \{(x, y, z) \in \mathbb{R}^3 : \ xyz = 1, \ x \geq 0, \ y \geq 0, \ z \geq 0\}.$$
   (a) Provare che $\sup_D f = +\infty$.
   (b) Determinare $\inf_D f$ specificando se si tratta di minimo.

2. Siano
$$D_1 := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}, \qquad D_2 := \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq y \leq 1/2\}.$$
   Calcolare
$$\int_{D_1} y \, dx \, dy, \qquad \int_{D_2} y \, dx \, dy.$$

3. Sia $B = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1, \ x \geq 0, \ y \geq 0\}$.
   (a) Mostrare che
$$\int_B \frac{y}{(x^2 + y^2)^2} dx \, dy = +\infty.$$
   (b) Stabilire se converge
$$\int_B \frac{y}{(x + y^2)^2} dx \, dy.$$

4. Sia γ la curva parametrizzata da $\gamma(t) = (t \sin t, t(\pi - t))$ con $0 \leq t \leq \pi$.
   (a) Provare che γ è chiusa e semplice e farne un disegno approssimativo.
   (b) Calcolare l'area del dominio racchiuso da γ.

Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato. Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. Sia $f(x,y,z) = x + y + z$ e sia D il dominio di $\mathbb{R}^3$ definito da

$$D = \{(x,y,z) \in \mathbb{R}^3 : \, xyz = 1, \, x \geq 0, \, y \geq 0, \, z \geq 0\}.$$

(a) Provare che $\sup_D f = +\infty$.

(b) Determinare $\inf_D f$ specificando se si tratta di minimo.

(a) $\lim\limits_{\substack{x \to +\infty \\ y = y_0 > 0}} f(x,y,z) = \lim\limits_{\substack{x \to +\infty \\ y = y_0 > 0}} x + y_0 + \frac{1}{xy_0} = +\infty \equiv \sup_D f$

(b) MODO 1 – SOSTITUZIONE DEL VINCOLO NELLA FUNZIONE

$xyz = 1 \quad z = \frac{1}{xy} \quad f(x,y,z) = g(x,y) = x + y + \frac{1}{xy}$

DOMINIO DI $g(x,y)$: $(x,y) \in \mathbb{R}^2 \quad x \neq 0 \quad y \neq 0 \equiv 1°$ Q ESCLUSI ASSI

($\equiv$ APERTO $\subseteq \mathbb{R}^2$)

COMPORTAMENTO AI LIMITI DEL DOMINIO:

$\lim\limits_{(x,y) \to +\infty} g(x,y) = +\infty \quad \lim\limits_{\substack{x \to 0^+ \\ y = y_0 > 0}} g(x,y) = \lim\limits_{\substack{y \to 0^+ \\ x = x_0 > 0}} g(x,y) = \lim\limits_{(x,y) \to (0^+,0^+)} g(x,y) = +\infty$

$\rightsquigarrow$ x WEIERSTRASS GENERAL. $\exists$ MINIMO ($\equiv \inf g = \inf_D f$)

PUNTI STAZIONARI:

$\begin{cases} g_x = 1 - \frac{y}{x^2y^2} = 1 - \frac{1}{x^2y} = 0 \\ g_y = 1 - \frac{x}{x^2y^2} = 1 - \frac{1}{xy^2} = 0 \end{cases} \xrightarrow{x^2y \neq 0} \begin{cases} x^2y = 1 \\ xy^2 = 1 \end{cases}$

P.TO DI MINIMO x WEIERSTRASS

$x^2y - xy^2 = 0 \rightsquigarrow xy(x-y) = 0 \rightsquigarrow \begin{cases} xy = 0 & \emptyset \\ y = x & \rightsquigarrow P_0 = (1,1) \end{cases}$

$\rightsquigarrow$ MIN $f(P_0) = 3 \equiv \inf_D f$

STUDIO PUNTI STAZIONARI (NON INDISPESABILE x WEIERSTRASS):

$g_{xx} = \frac{2}{x^3y} \qquad g_{yy} = \frac{2}{xy^3} \qquad g_{xy} = g_{yx} = \frac{1}{x^2y^2}$

$Hg(P_0) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \det Hg(P) = 4 - 1 = 3 \rightsquigarrow P_1$ PTO DI MINIMO

$\rightsquigarrow \inf_D f = f(1,1,1) = 3$ ANCHE MINIMO DI f IN D

MODO 2 - MOLTIPLICATORI DI LAGRANGE

$$\Phi(x,y,z) = xyz - 1 = 0$$

SISTEMA 1

$$\left.\begin{cases} \left.\begin{aligned} &\Phi_x = yz = 0 \\ &\Phi_y = xz = 0 \\ &\Phi_z = xy = 0 \end{aligned}\right\} \rightsquigarrow x = y = z = 0 \\ \Phi = xyz - 1 = 0 \rightsquigarrow 0 = 1 \end{cases}\right\} \rightsquigarrow \emptyset$$

SISTEMA 2

$$\begin{cases} \left.\begin{aligned} &f_x = 1 = \lambda yz \\ &f_y = 1 = \lambda xz \\ &f_z = 1 = \lambda xy \end{aligned}\right\} \lambda^3 (xyz)^2 = 1 \rightsquigarrow \lambda^3 = 1 \quad \lambda = 1 \\ \Phi = xyz - 1 = 0 \end{cases}$$

$$\begin{cases} \left.\begin{aligned} &yz = 1 \\ &xz = 1 \end{aligned}\right\} z(y-x) = 0 \begin{cases} z = 0 \rightsquigarrow \emptyset \\ y = x \end{cases} \\ \left.\begin{aligned} &xy = 1 \rightsquigarrow x^2 = 1 \quad x = 1 \geq 0 \quad y = 1 \\ &xyz = 1 \rightsquigarrow z = 1 \end{aligned}\right\} P_0 = (1,1,1) \end{cases}$$

P.TO DI MINIMO x WEIERSTRASS

$\rightsquigarrow$ INF $f = f(1,1,1) = 3$ ANCHE MINIMO DI f IN D

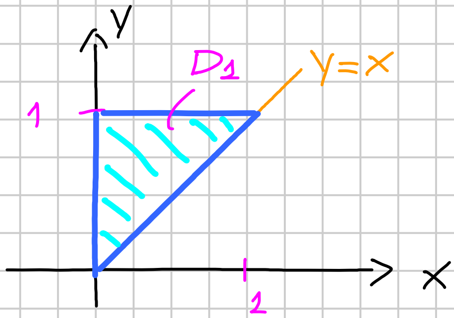
2. Siano

$$D_1 := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\}, \qquad D_2 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le y \le 1/2\}.$$

Calcolare

$$\int_{D_1} y\,dx\,dy, \qquad \int_{D_2} y\,dx\,dy.$$

$$\int_{D_2} y\,dx\,dy \left[\equiv y_G \cdot AREA(D_1) = \frac{2}{3}\cdot\frac{1}{2} = 1/3\right]$$



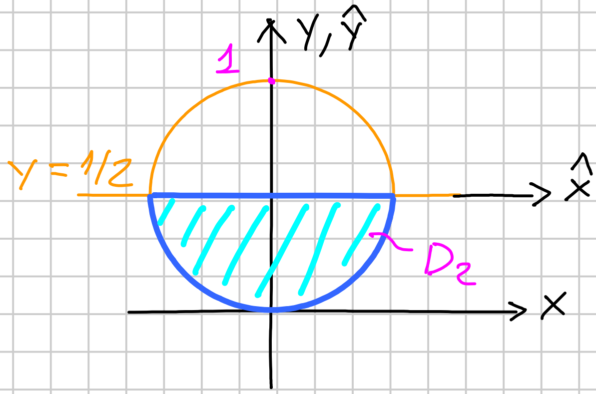
$$= \int_0^1 \int_x^1 y\,dy\,dx = \int_0^1 \left[y^2/2\right]_x^1 dx =$$

$$= \int_0^1 \frac{1}{2} - \frac{x^2}{2}\,dx = \left[\frac{1}{2}x - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

$$D_2 : \begin{cases} y \le 1/2 \\ x^2+y^2 \le y \end{cases} \qquad x^2+y^2-y=0 \Downarrow \qquad \begin{cases} (x-x_0)^2+(y-y_0)^2 = R^2 \\ x^2+y^2-2x_0x-2y_0y = R^2-x_0^2-y_0^2 \end{cases}$$

CIRCONF. CON CENTRO $(x_0,y_0)=(0,\frac{1}{2})$ $R=\frac{1}{2}$

$$\int_{D_2} y\,dx\,dy \left[\equiv y_G \cdot AREA(D_2)\right]$$



$$= \int_{D_2} \left(\hat{y}+\frac{1}{2}\right) dx\,dy = \int_{\pi}^{2\pi}\int_0^{1/2} \left(\rho^2 \sin\theta + \frac{\rho}{2}\right) d\rho\,d\theta =$$

$$= \int_{\pi}^{2\pi} \sin\theta \left[\rho^3/3\right]_0^{1/2} d\theta + \int_{\pi}^{2\pi} \left[\rho^2/4\right]_0^{1/2} d\theta = \frac{1}{24}\left[-\cos\theta\right]_{\pi}^{2\pi} + \frac{\pi}{16} = \frac{\pi}{16} - \frac{1}{12}$$

VERIFICA

$$2\pi d \cdot A = \frac{4}{3}\pi R^3 \qquad d = \frac{4}{3}\cancel{\pi} R^{\cancel{3}} \frac{1}{\cancel{2\pi}} \frac{\cancel{2}}{\pi \cancel{R^2}} = \frac{4}{3}\frac{R}{\pi} = \frac{2}{3\pi}$$

$$\rightsquigarrow y_G = \frac{1}{2} - d = \frac{1}{2} - \frac{2}{3\pi} \qquad y_G \cdot A = \frac{\pi(1/2)^2}{2}\left(\frac{1}{2} - \frac{2}{3\pi}\right) =$$

$$= \frac{\pi}{8}\left(\frac{1}{2} - \frac{2}{3\pi}\right) = \frac{\pi}{16} - \frac{1}{12}$$

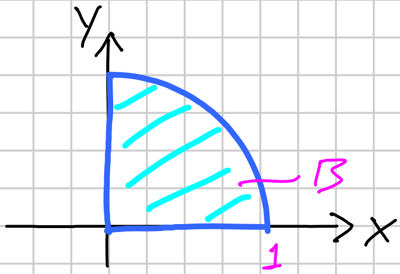
3. Sia $B = \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

(a) Mostrare che
$$\int_B \frac{y}{(x^2+y^2)^2}\,dx\,dy = +\infty.$$

(b) Stabilire se converge
$$\int_B \frac{y}{(x+y^2)^2}\,dx\,dy.$$

(a) $$\int_B \frac{y}{(x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 \frac{\rho \sin\theta}{\rho^4}\,\rho\,d\rho\,d\theta =$$

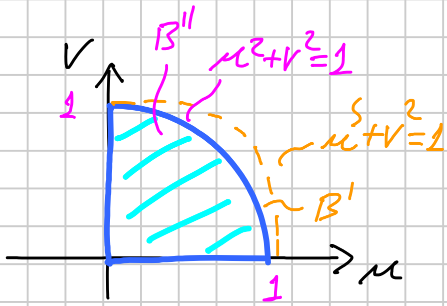
$$= \int_0^{\pi/2}\int_0^1 \frac{\sin\theta}{\rho^2}\,d\rho\,d\theta \ge \int_{\pi/6}^{\pi/2}\int_0^1 \frac{\sin\theta}{\rho^2}\,d\rho\,d\theta$$



$$\ge \sin\frac{\pi}{6}\int_0^1 \frac{1}{\rho^2}\,d\rho = +\infty$$ DIVERGE

(b) $$\int_B \frac{y}{(x+y^2)^2}\,dx\,dy \qquad u^2=x \quad v=y \qquad x^2+y^2=1 \rightsquigarrow u^4+v^2=1$$

$$= \int_{B'} \frac{v}{(u^2+v^2)^2}\,2u\,du\,dv \ge \int_{B''} \frac{2uv}{(u^2+v^2)^2}\,du\,dv$$



$$\begin{cases} u = \rho\cos\theta \\ v = \rho\sin\theta \end{cases} \quad = \int_0^{\pi/2}\int_0^1 \frac{\rho^2 \sin 2\theta}{\rho^4}\,\rho\,d\rho\,d\theta = \int_0^{\pi/2}\int_0^1 \frac{\sin 2\theta}{\rho}\,d\rho\,d\theta$$

$$\ge \int_{\pi/8}^{\pi/4}\int_0^1 \frac{\sin 2\theta}{\rho}\,d\rho\,d\theta \ge \frac{\sqrt{2}}{2}\int_0^1 \frac{1}{\rho}\,d\rho = +\infty$$ DIVERGE

4. Sia γ la curva parametrizzata da $\gamma(t) = (t \sin t, t(\pi - t))$ con $0 \le t \le \pi$.

(a) Provare che γ è chiusa e semplice e farne un disegno approssimativo.

(b) Calcolare l'area del dominio racchiuso da γ.

(a) $\gamma(t) = (t \operatorname{sen} t,\ t(\pi - t))$

$\gamma(0) = (0,0) = \gamma(\pi) \rightsquigarrow \gamma$ È CHIUSA

SIANO $t, s \in [0, \pi]$

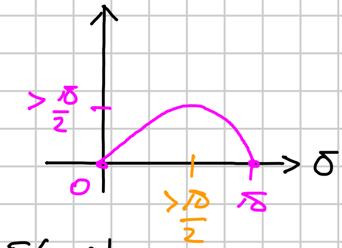
$$\begin{cases} t \operatorname{sen} t = s \operatorname{sen} s \\ t(\pi - t) = s(\pi - s) \end{cases} \quad \begin{cases} t \operatorname{sen} t = s \operatorname{sen} s \\ t\pi - t^2 = s\pi - s^2 \end{cases} \quad \begin{cases} t \operatorname{sen} t = s \operatorname{sen} s \\ \pi(t - s) = t^2 - s^2 \end{cases} \quad t \neq s \rightsquigarrow$$

$$\begin{cases} t \operatorname{sen} t = (\pi - t)\operatorname{sen}(\pi - t) = (\pi - t)\operatorname{sen} t \Rightarrow t = \pi - t \\ t + s = \pi \quad s = \pi - t \end{cases} \quad \begin{cases} t = \pi/2 \\ s = \pi - t = \pi/2 \end{cases}$$

$\rightsquigarrow \gamma(t) = \gamma(s) \Leftrightarrow s = t \rightsquigarrow \gamma$ È SEMPLICE

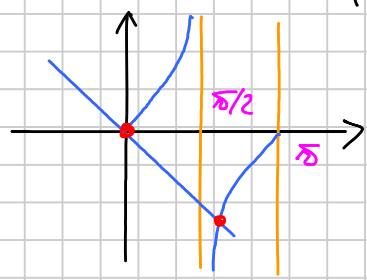
DISEGNO DI γ

$t \operatorname{sen} t$

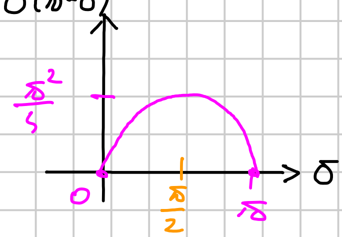


$(t \operatorname{sen} t)' = \operatorname{sen} t + t \cos t = 0 \qquad \cos t(\operatorname{tg} t + t) = 0$

$\cos t \neq 0 \qquad \operatorname{tg} t = -t$



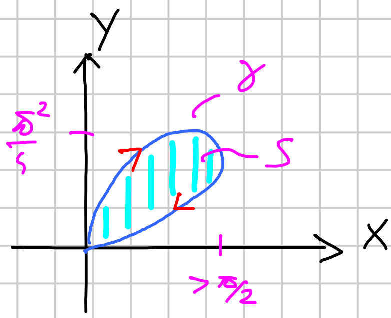
$t(\pi - t)$



$[t(\pi - t)]' = \pi - t - t = 0$

$t = \pi/2$

$\rightsquigarrow$



(b) GAUSS-GREEN $\vec{E}=(x,0) \rightsquigarrow$ AREA $= \int_{\partial S} x\,dy$

$$x(\sigma)=\sigma \operatorname{sen}\sigma \qquad y(\sigma)=\sigma(\pi-\sigma) \qquad dy=(\pi-2\sigma)\,d\sigma$$

<u>OSS</u> SEGNO DI PERCORRENZA ORARIO $\rightsquigarrow$ AGGIUNGO SEGNO "−"

$$-\int_{\partial S} x\,dy = -\int_0^{\pi} \sigma \operatorname{sen}\sigma\,(\pi-2\sigma)\,d\sigma = -\int_0^{\pi}\left(\pi\sigma\operatorname{sen}\sigma - 2\sigma^2\operatorname{sen}\sigma\right)d\sigma$$

$$= -\pi\int_0^{\pi}\sigma\operatorname{sen}\sigma\,d\sigma + 2\int_0^{\pi}\sigma^2\operatorname{sen}\sigma\,d\sigma$$

$$\int_0^{\pi} \overset{f}{\sigma}\,\overset{g'}{\operatorname{sen}\sigma}\,d\sigma = \Big[-\sigma\cos\sigma\Big]_0^{\pi} + \int_0^{\pi}\cos\sigma\,d\sigma = -\pi(-1) + \Big[\operatorname{sen}\sigma\Big]_0^{\pi} = \pi$$

$$\int_0^{\pi} \overset{f}{\sigma^2}\,\overset{g'}{\operatorname{sen}\sigma}\,d\sigma = \Big[-\sigma^2\cos\sigma\Big]_0^{\pi} + \int_0^{\pi}\overset{\ell}{2\sigma}\,\overset{m'}{\cos\sigma}\,d\sigma =$$

$$= -\pi^2(-1) + \Big[2\sigma\operatorname{sen}\sigma\Big]_0^{\pi} - \int_0^{\pi} 2\operatorname{sen}\sigma\,d\sigma =$$

$$= \pi^2 - 2\Big[-\cos\sigma\Big]_0^{\pi} = \pi^2 - 4$$

$$\rightsquigarrow -\pi\int_0^{\pi}\sigma\operatorname{sen}\sigma\,d\sigma + 2\int_0^{\pi}\sigma^2\operatorname{sen}\sigma\,d\sigma = -\pi^2 + 2(\pi^2-4) = \pi^2 - 8$$

$\rightsquigarrow$ AREA $= \pi^2 - 8$