Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Scritto d'esame di Analisi Matematica II

Pisa, ?? ?? ????

1. Si consideri l'insieme D definito da

$$D := \{(x, y, z) \in \mathbb{R}^3 : \; x + y + z = 0, \, x^2 + y = 1, \, y \geq 0\}.$$

   (a) Provare che D è limitato.

   (b) Determinare estremo superiore e inferiore di $f(x, y, z) = xy + z$ su D precisando se si tratta di massimo e/o minimo ed in caso affermativo determinare anche i punti di massimo/minimo.

2. Siano:

$$B_1 := \{(x, y) \in \mathbb{R}^2 : \, x^2+y^2 \leq 1\}, \qquad B_2 := \{(x, y) \in \mathbb{R}^2 : \, x^2+y^2 \leq 1, \, x \geq 1/2\}.$$

   Calcolare

$$\int_{B_1} |y| \sqrt{x^2 + y^2}\, dx\, dy, \qquad \int_{B_2} |y| \sqrt{x^2 + y^2}\, dx\, dy.$$

3. Sia $D := [1, +\infty[\times [0, 1]$. Stabilire per quali $\alpha > 0$ si ha

$$\int_D \frac{x^\alpha}{1 + x^2 + xy^2} dx\, dy < +\infty.$$

4. Si considerino il campo di vettori F e la superficie S definiti da

$$F(x, y, z) = (x+y^4, 2y-z^3, x^2-3z), \qquad S := \{(x, y, z) \in \mathbb{R}^3 : \, x^2+y^2+z^4 = 1, \, z \geq 0\}.$$

   Si supponga che S sia orientata prendendo in $(0, 0, 1)$ la normale in direzione $(0, 0, 1)$. Calcolare il flusso di F attraverso S.

> Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato.
> Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. Si consideri l'insieme D definito da

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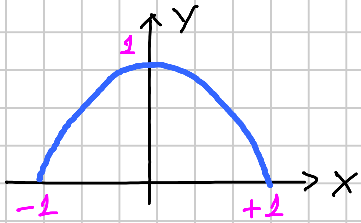
(a) Provare che D è limitato.

(b) Determinare estremo superiore e inferiore di $f(x,y,z) = xy + z$ su D precisando se si tratta di massimo e/o minimo ed in caso affermativo determinare anche i punti di massimo/minimo.

(a)

$$\begin{cases} y = 1-x^2 \\ y \geq 0 \end{cases} \leadsto \begin{cases} -1 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$$



$$\begin{cases} z = -x-y = -x-1+x^2 & z'(x) = -1+2x = 0 \quad x = 1/2 \\ z(1/2) = -1/2 - 1 + 1/4 = -5/4 & z(1) = -1 \quad z(-1) = 1 \end{cases} \leadsto -\frac{5}{4} \leq z \leq 1$$

(b) D É COMPATTO ($\equiv$ CHIUSO E LIMITATO) $\leadsto$ X TEOR. DI WEIERSTRASS ESISTONO MAX E MIN

MODO 1 "= ANALISI 1"

$f(x,y,z) = xy + z \qquad y = 1-x^2 \geq 0 \qquad z = -x-y = -x-1+x^2$

$$g(x) = f(x, y(x), z(x)) = x(1-x^2) - x - 1 + x^2 = \cancel{x} - x^3 - \cancel{x} - 1 + x^2 =$$
$$= -x^3 + x^2 - 1 \qquad -1 \leq x \leq 1$$

$$g'(x) = -3x^2 + 2x = 0 \quad x(-3x+2) = 0 \quad \begin{cases} x = 0 & g(0) = -1 \quad P_1 \\ x = 2/3 & g(2/3) = -\frac{8}{27} + \frac{4}{9} - 1 \end{cases}$$

$$= \frac{-8+12-27}{27} = \frac{-23}{27} \quad P_2$$

$$g''(x) = -6x + 2 \quad \begin{cases} g''(0) = 2 & \leadsto \text{MIN} \\ g''(2/3) = -6 \cdot 2/3 + 2 = -2 & \leadsto \text{MAX} \end{cases}$$

$$g(-1) = 1 + 1 - 1 = 1 \qquad g(1) = -1 + 1 - 1 = -1$$

$$\leadsto \begin{cases} \text{SUP}(f) = 1 \equiv \text{MAX} f \quad \text{IN} \quad P_1 = (-1, 0, 1) \\ \text{INF}(f) = -1 \equiv \text{MIN} f \quad \text{IN} \quad P_3 = (0, 1, -1) \ \text{E} \ P_4 = (1, 0, -1) \end{cases}$$

MODO 2 "MOLT. DI LAGRANGE"

$f(x,y,z) = xy + z \qquad \Phi = x + y + z = 0 \qquad \Psi = x^2 + y - 1 = 0$

$$\text{SISTEMA 1: } \begin{cases} \begin{array}{c} \nabla\Phi \\ \nabla\Psi \end{array} \begin{vmatrix} 1 & \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} \\ 2x & \end{vmatrix} \quad \text{DET} = -1 \neq 0 \\ \Phi = 0, \ \Psi = 0 \end{cases} \rightsquigarrow \text{NON CI SONO PUNTI SINGOLARI}$$

SISTEMA 2:

$$\begin{cases} f_x = \lambda \Phi_x + \mu \Psi_x \\ f_y = \lambda \Phi_y + \mu \Psi_y \\ f_z = \lambda \Phi_z + \mu \Psi_z \\ \Phi = 0 \quad \Psi = 0 \end{cases} \qquad \begin{cases} y = \lambda + 2\mu x \\ x = \lambda + \mu \rightsquigarrow \mu = x - 1 \\ 1 = \lambda \quad \rightsquigarrow \lambda = 1 \\ \Phi = 0 \quad \Psi = 0 \end{cases}$$

$$\begin{cases} y = 1 + 2x(x-1) = 1 + 2x^2 - 2x \\ \Phi = 0 \quad \Psi = 0 \end{cases} \qquad \begin{cases} 1 + 2x^2 - 2x = 1 - x^2 \\ \Phi = 0 \quad \Psi = 0 \end{cases}$$

$$3x^2 - 2x = 0 \rightsquigarrow x(3x-2) = 0 \qquad \begin{cases} x = 0 \rightsquigarrow y = 1 \quad z = -1 \\ x = 2/3 \rightsquigarrow y = 1 - \frac{4}{9} = \frac{5}{9} \end{cases}$$

$$\rightsquigarrow \begin{cases} P_1 = (0, 1, -1) \\ P_2 = \left(\frac{2}{3}, \frac{5}{9}, -\frac{11}{9}\right) \end{cases} \qquad z = -\frac{2}{3} - \frac{5}{9} = \frac{-6-5}{9} = -\frac{11}{9}$$

$$\begin{cases} f(P_1) = -1 \qquad f(P_2) = \frac{2}{3} \cdot \frac{5}{9} - \frac{11}{9} = \frac{10 - 33}{27} = \frac{-23}{27} \\ f(P_3) = f(-1, 0, 1) = 1 \qquad f(P_4) = (1, 0, -1) = -1 \end{cases}$$

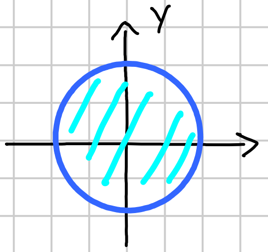
$$\rightsquigarrow \begin{cases} \sup(f) = 1 \equiv \max f \quad \text{IN} \quad P_1 = (-1, 0, 1) \\ \inf(f) = -1 \equiv \min f \quad \text{IN} \quad P_3 = (0, 1, -1) \ \text{E} \ P_5 = (1, 0, -1) \end{cases}$$

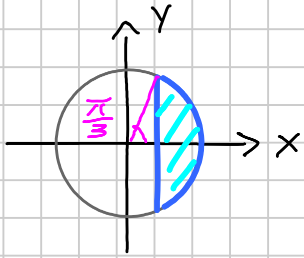
2. Siano:

$$B_1 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}, \qquad B_2 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1,\ x \ge 1/2\}.$$

Calcolare

$$\int_{B_1} |y|\sqrt{x^2+y^2}\,dx\,dy, \qquad \int_{B_2} |y|\sqrt{x^2+y^2}\,dx\,dy.$$



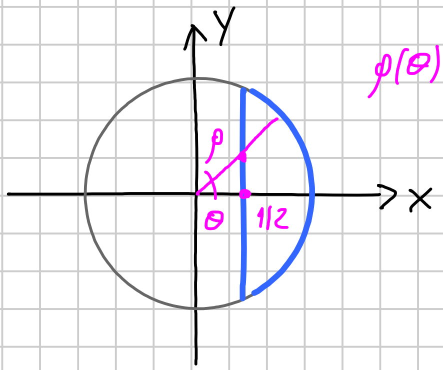


$$\int_{B_2} |y|\sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 |\rho\sin\theta|\,\rho^2\,d\rho\,d\theta = 2\int_0^{\pi}\int_0^1 \rho^3\sin\theta\,d\rho\,d\theta =$$

$$= 2\int_0^{\pi} \sin\theta\,\left[\rho^4/4\right]_0^1 d\theta = \frac{1}{2}\left[-\cos\theta\right]_0^{\pi} = 1$$

$$\int_{B_2} |y|\sqrt{x^2+y^2}\,dx\,dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{\rho(\theta)}^1 |\rho\sin\theta|\,\rho^2\,d\rho\,d\theta = 2\int_0^{\pi/3}\int_{\rho(\theta)}^1 \rho^3\sin\theta\,d\rho\,d\theta =$$

$$= 2\int_0^{\pi/3} \sin\theta\left[\frac{\rho^4}{4}\right]_{1/2\cos\theta}^{1} d\rho\,d\theta =$$



$$\rho(\theta) = \frac{1/2}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\pi/3} \sin\theta\left(1 - \frac{1}{16\cos^4\theta}\right) d\theta =$$

$$= \frac{1}{2}\int_0^{\pi/3} \sin\theta\,d\theta - \frac{1}{32}\int_0^{\pi/3} \frac{\sin\theta}{\cos^4\theta}\,d\theta = \frac{1}{2}\left[-\cos\theta\right]_0^{\pi/3} + \frac{1}{32}\int_0^{\pi/3} \frac{d(\cos\theta)}{\cos^4\theta} =$$

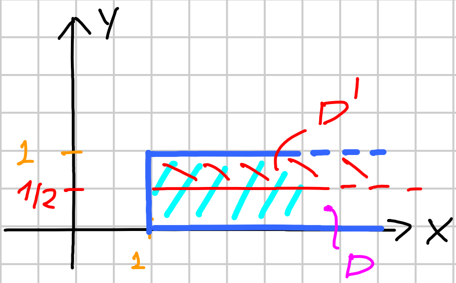
$$= \frac{1}{2}\left(-\frac{1}{2}+1\right) + \frac{1}{32}\left[-\frac{1}{3\cos^3\theta}\right]_0^{\pi/3} = \frac{1}{4} + \frac{1}{32}\left(-\frac{1}{3}\cdot 8 + \frac{1}{3}\right) =$$

$$= \frac{1}{4} + \frac{1}{32}\left(-\frac{7}{3}\right) = \frac{1}{4} - \frac{7}{96} = \frac{24-7}{96} = \frac{17}{96}$$

3. Sia $D := [1, +\infty[\times[0,1]$. Stabilire per quali $\alpha > 0$ si ha

$$\int_D \frac{x^\alpha}{1+x^2+xy^2}\,dx\,dy < +\infty.$$

$$\int_D \frac{x^\alpha}{1+x^2+xy^2}\,dx\,dy \le \int_D \frac{x^\alpha}{x^2}\,dx\,dy =$$



$$= 1 \cdot \int_1^{+\infty} \frac{1}{x^{2-\alpha}}\,dx < +\infty \qquad 2-\alpha > 1 \rightarrow \alpha < 1$$

PER $\alpha \geq 1$ $\displaystyle\int_D \frac{x^\alpha}{1+x^2+xy^2}\,dx\,dy \geq \int_{D'} \frac{x^\alpha}{1+x^2+xy^2}\,dx\,dy =$

$$= \int_{1/2}^{1}\int_1^{+\infty} \frac{x^\alpha}{1+x^2+xy^2}\,dx\,dy \geq \frac{1}{2}\int_1^{+\infty} \frac{x^\alpha}{1+x^2+\frac{1}{5}x}\,dx$$

$$x \to +\infty \qquad \frac{x^\alpha}{1+x^2+\frac{1}{5}x} \sim \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}}$$

$\rightarrow$ STESSO COMPORTAMENTO DI $\displaystyle\int_1^{+\infty} \frac{1}{x^{2-\alpha}}\,dx = +\infty \qquad \alpha \geq 1$

4. Si considerino il campo di vettori F e la superficie S definiti da

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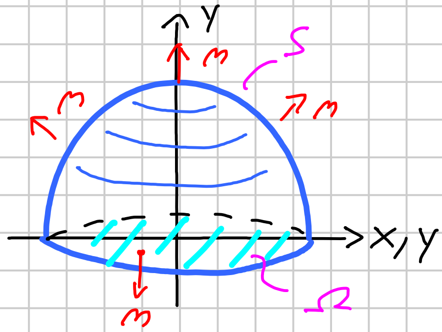
Si supponga che S sia orientata prendendo in $(0,0,1)$ la normale in direzione $(0,0,1)$. Calcolare il flusso di F attraverso S.

$$\int_S \vec{F}\cdot\vec{n}\, d\sigma$$



$$\int_V \mathrm{div}\,\vec{F}\, dV = \int_S \vec{F}\cdot\vec{n}\, d\sigma + \int_\Omega \vec{F}\cdot\vec{n}\, d\sigma$$

$$\rightsquigarrow \int_S \vec{F}\cdot\vec{n}\, d\sigma = \int_V \mathrm{div}\,\vec{F}\, dV - \int_\Omega \vec{F}\cdot\vec{n}\, d\sigma \qquad \Omega = \{(u,v,0) : u^2+v^2 \leq 1\}$$

$$\mathrm{div}\,\vec{F} = A_x + B_y + C_z = 1 + 2 - 3 = 0 \qquad \rightsquigarrow \int_V \mathrm{div}\,\vec{F}\, dV = 0$$

$$\int_\Omega \vec{F}\cdot\vec{n}\, d\sigma = \int_\Omega (3z - x^2)\, d\sigma = -\int_\Omega x^2\, d\sigma =$$

$$= -\int_0^{2\pi}\int_0^1 \rho^3 \cos^2\theta\, d\rho\, d\theta = -\int_0^{2\pi} \cos^2\theta \left[\rho^4/4\right]_0^1 d\theta =$$

$$= -\frac{1}{4}\int_0^{2\pi} \cos^2\theta\, d\theta = -\frac{\pi}{4}$$

$$\rightsquigarrow \int_S \vec{F}\cdot\vec{n}\, d\sigma = \int_V \mathrm{div}\,\vec{F}\, dV - \int_\Omega \vec{F}\cdot\vec{n}\, d\sigma = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$