3. Sia $D := [0,+\infty[\times[0,+\infty[$.

(a) Provare che
$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^4}\,dx\,dy < +\infty.$$

(b) Stabilire per quali $\alpha > 0$ converge
$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy.$$

(a)
$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^4}\,dx\,dy = \int_0^{\pi/2}\int_0^{+\infty} \frac{\log(1+\rho^2\sin\theta\cos\theta)}{1+\rho^8}\,\rho\,d\rho\,d\theta =$$
$$\leq \int_0^{\pi/2}\int_0^{+\infty} \frac{\log(1+\rho^2)}{1+\rho^8}\,\rho\,d\rho\,d\theta = \frac{\pi}{2}\int_0^{+\infty}\frac{\log(1+\rho^2)}{1+\rho^8}\,\rho\,d\rho \leq$$

NON INDEFINITO CONVERGE — CONVERGE

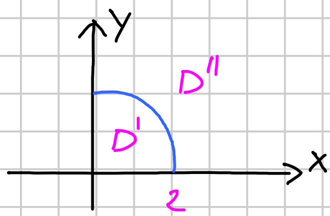
$$\leq \frac{\pi}{2}\int_0^1 \frac{\log(1+\rho^2)}{1+\rho^8}\,\rho\,d\rho + \frac{\pi}{2}\int_1^{+\infty}\frac{\rho^2}{\rho^8}\,\rho\,d\rho < +\infty \qquad \begin{cases}\log(1+x)\leq x\\ 1+x > x\end{cases}$$

(b)

NON INDEFINITO CONVERGE $\forall\alpha$ — INDEFINITO

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy = \int_{D'} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy + \int_{D''} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy$$

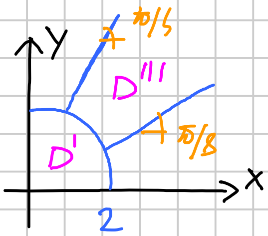
$$\int_{D''} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy = \int_0^{\pi/2}\int_2^{+\infty}\frac{\log(1+\frac{1}{2}\rho^2\sin 2\theta)}{1+\rho^{2\alpha}}\,\rho\,d\rho\,d\theta$$




$$\leq \frac{\pi}{2}\int_2^{+\infty}\frac{\log(1+\frac{1}{2}\rho^2)\,\rho}{1+\rho^{2\alpha}}\,d\rho < +\infty \quad \text{PER } 2\alpha-1>1 \leadsto \alpha>1$$

INFATTI: $\rho\to+\infty \quad \dfrac{\log(1+\frac{1}{2}\rho^2)\,\rho}{1+\rho^{2\alpha}} \sim \dfrac{\log\rho}{\rho^{2\alpha-1}} = \dfrac{1}{\rho^{2\alpha-1}(\log\rho)^{-1}}$

PER $\alpha\leq 1$




$$\int_{D''}\frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy \geq \int_{D'''}\frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy =$$
$$= \int_{\pi/8}^{\pi/4}\int_2^{+\infty}\frac{\log(1+\frac{1}{2}\rho^2\sin 2\theta)}{1+\rho^{2\alpha}}\,\rho\,d\rho\,d\theta \geq \frac{\pi}{8}\int_2^{+\infty}\frac{\log(1+\frac{\sqrt{2}}{4}\rho^2)\,\rho}{1+\rho^{2\alpha}}\,d\rho$$
$$\geq \frac{\pi}{8}\log(2+\sqrt{2})\int_2^{+\infty}\frac{\rho}{1+\rho^{2\alpha}}\,d\rho = +\infty \quad \text{PER } 2\alpha-1\leq 1 \leadsto \alpha\leq 1$$