BY GIMUSI 01/06/14

Università di Pisa - Corso di Laurea in Ingegneria Meccanica

Scritto d'esame di Analisi Matematica II

Pisa, ?? ?? ????

1. Sia $f(x,y) = x^2y$ e sia D il dominio di $\mathbb{R}^2$ definito da

$$D = \{(x,y) \in \mathbb{R}^2 : \; x^4 + y^4 \leq 27, \, x \geq 0, \, y \geq 0\}.$$

Determinare massimo e minimo di f in D e i relativi punti di massimo/minimo.

2. Sia V il solido di $\mathbb{R}^3$ definito da

$$V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^4 \leq 2, \, x^2 + y^2 \leq z^4, \, z \geq 0\}.$$

   (a) Calcolare il volume di V.

   (b) Calcolare

$$\int_V x^2 \, dx \, dy \, dz.$$

3. Sia $D := [0, +\infty[\times [0, +\infty[$.

   (a) Provare che

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^4} dx \, dy < +\infty.$$

   (b) Stabilire per quali $\alpha > 0$ converge

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha} dx \, dy.$$

4. Sia D il dominio di $\mathbb{R}^2$ delimitato dalla curva γ parametrizzata da $\gamma(t) = (t - t^2, t^3)$ con $0 \leq t \leq 1$ e dall'asse delle y.

   (a) Fare un disegno approssimativo di D.

   (b) Calcolare

$$\int_D y^3 dx \, dy.$$

Si ricorda che ogni passaggio deve essere *adeguatamente* giustificato. Ogni esercizio verrà valutato in base alla *correttezza* ed alla *chiarezza* delle spiegazioni fornite. La sola scrittura del risultato non ha alcun valore.

1. Sia $f(x,y) = x^2y$ e sia D il dominio di $\mathbb{R}^2$ definito da

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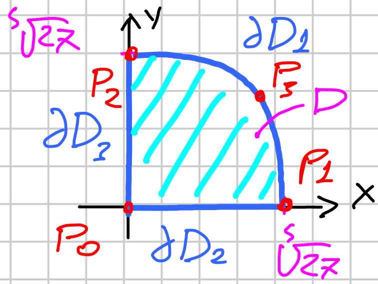
$f(x,y) = x^2 y \qquad D: \; x^4 + y^4 \leq 27, \; x \geq 0, \; y \geq 0$



D COMPATTO $\rightsquigarrow$ MAX/MIN ESISTONO

1) PUNTI SINGOLARI INTERNI $\rightsquigarrow$ NESSUNO

2) PUNTI STAZIONARI INTERNI

$$\begin{cases} f_x = 2xy = 0 \\ f_y = x^2 = 0 \end{cases} \quad \begin{cases} \forall y \\ x = 0 \end{cases} \rightsquigarrow \text{ASSE } y \subseteq \partial D$$

3) STUDIO SUL BORDO $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$

<u>BORDO ∂D_1</u> $\quad \Phi(x,y) = x^4 + y^4 - 27 = 0$

SISTEMA 1 $\begin{cases} \Phi_x = 4x^3 = 0 \\ \Phi_y = 4y^3 = 0 \\ \Phi = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \\ -27 = 0 \end{cases} \rightsquigarrow \emptyset$

SISTEMA 2 $\begin{cases} f_x = \lambda \Phi_x \\ f_y = \lambda \Phi_y \\ \Phi = 0 \end{cases} \rightsquigarrow \begin{cases} 2xy = \lambda \cdot 4x^3 \\ x^2 = \lambda \cdot 4y^3 \\ x^4 + y^4 - 27 = 0 \end{cases} \rightsquigarrow$

$$\begin{cases} 2xy \cdot y^3 = \lambda 4x^3 y^3 \\ x^2 \cdot x^3 = \lambda 4 y^3 x^3 \end{cases} \rightsquigarrow 2xy^4 = x^5 \qquad 2xy^4 - x^5 = 0$$

$$x(2y^4 - x^4) = 0 \quad \begin{cases} x = 0 \rightsquigarrow y^4 = 27 \quad y = \begin{cases} \sqrt[4]{27} \rightsquigarrow P_2 \\ -\sqrt[4]{27} \rightsquigarrow \emptyset \end{cases} \\ 2y^4 - x^4 = 0 \rightsquigarrow 3y^4 = 27 \quad y = +\sqrt{3}, x = \sqrt[4]{18} \\ \qquad \rightsquigarrow P_3 = (\sqrt[4]{18}, \sqrt{3}) \\ x = 0, \; 2y^4 = 0 \quad y = 0 \rightsquigarrow P_0 \notin \partial D_1 \end{cases}$$

$\rightsquigarrow f(P_2) = 0 \qquad f(P_3) = (\sqrt[4]{18})^2 \sqrt{3} = \sqrt{18} \cdot \sqrt{3} = 3\sqrt{6}$

BORDO ∂D_2 $\quad \Phi(x,y) = y = 0 \quad \rightsquigarrow \quad f(x,0) = 0$

BORDO ∂D_3 $\quad \Phi(x,y) = x = 0 \quad \rightsquigarrow \quad f(0,y) = 0$

$$\rightsquigarrow \begin{cases} \text{MAX } f = 3\sqrt{6} & \text{IN } P_3 = (\sqrt[5]{18}, \sqrt{3}) \\ \text{MIN } f = 0 & \text{IN } P \in \partial D_2 \cup \partial D_3 \end{cases}$$

2. Sia V il solido di $\mathbb{R}^3$ definito da

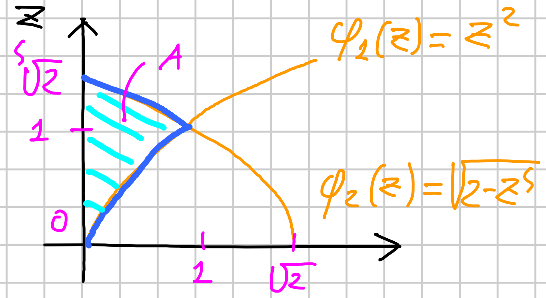
$$V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^4 \leq 2,\ x^2 + y^2 \leq z^4,\ z \geq 0\}.$$

(a) Calcolare il volume di V.

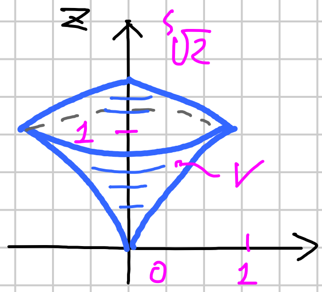
(b) Calcolare

$$\int_V x^2 \, dx\, dy\, dz.$$

$$\begin{cases} \varphi_1(z) = z^2 \quad \varphi_2(z) = \sqrt{2 - z^5} \\ \varphi_1(z) = \varphi_2(z) \quad z^2 = \sqrt{2 - z^5} \quad z^5 = 2 - z^5 \quad z = 1 \\ \varphi_2(z) = 0 \quad z^5 = 2 \quad z = \sqrt[5]{2} \end{cases}$$



(a) VOLUME DI V



MODO 1 $\quad V = \int_0^1 \pi \varphi_1(z)^2 \, dz + \int_1^{\sqrt[5]{2}} \pi \varphi_2(z)^2 \, dz =$

$$= \pi \int_0^1 z^5 \, dz + \pi \int_1^{\sqrt[5]{2}} (2 - z^5) \, dz =$$

$$= \pi \left[\frac{z^5}{5} \right]_0^1 + \pi \left[2z - \frac{z^5}{5} \right]_1^{\sqrt[5]{2}} = \frac{\pi}{5} + \pi \left(2\sqrt[5]{2} - \frac{2\sqrt[5]{2}}{5} - 2 + \frac{1}{5} \right) =$$

$$= \frac{\pi}{5} + \pi \frac{8}{5}\sqrt[5]{2} - \frac{8}{5}\pi = \pi \frac{8}{5}\sqrt[5]{2} - \frac{8}{5}\pi = \frac{8}{5}\pi(\sqrt[5]{2} - 1)$$

MODO 2 $\quad$ GULDINO: $\quad V = 2\pi y_G \cdot A$

$$y_G \cdot A = \int_0^1 \frac{1}{2}\varphi_1^2(z)\,dz + \int_1^{\sqrt[5]{2}} \frac{1}{2}\varphi_2^2(z)\,dz =$$

$$= \frac{1}{2}\left(\frac{8}{5}(\sqrt[5]{2}-1)\right) \rightsquigarrow V = \frac{8}{5}\pi(\sqrt[5]{2}-1)$$

(b) $\int_V x^2\,dx\,dy\,dz =$ COORDINATE CILINDRICHE $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$

$$= \int_0^{2\pi}\int_0^1\int_0^{z^2} \rho^3\cos^2\theta\,d\rho\,dz\,d\theta + \int_0^{2\pi}\int_1^{\sqrt[5]{2}}\int_0^{\sqrt{2-z^5}} \rho^3\cos^2\theta\,d\rho\,dz\,d\theta$$

$$\int_0^{2\pi}\int_0^1\int_0^{z^2} \rho^3\cos^2\theta\,d\rho\,dz\,d\theta = \int_0^{2\pi}\int_0^1 \cos^2\theta\left[\frac{\rho^4}{4}\right]_0^{z^2} dz\,d\theta =$$

$$= \int_0^{2\pi}\int_0^1 \cos^2\theta\,\frac{z^8}{4}\,dz\,d\theta = \int_0^{2\pi}\cos^2\theta\left[\frac{z^9}{36}\right]_0^1 d\theta = \frac{1}{36}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{\pi}{36}$$

$$\int_0^{2\pi}\int_1^{\sqrt[5]{2}}\int_0^{\sqrt{2-z^5}} \rho^3\cos^2\theta\,d\rho\,dz\,d\theta = \pi\int_1^{\sqrt[5]{2}}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{2-z^5}} dz = \frac{\pi}{4}\int_1^{\sqrt[5]{2}}(2-z^5)^2\,dz =$$

$$= \frac{\pi}{4}\left[\frac{z^{11}}{11} - \frac{4}{6}z^6 + 4z\right]_1^{\sqrt[5]{2}} = \frac{\pi}{4}\left(\frac{4}{11}\sqrt[5]{2} - \frac{8}{6}\sqrt[5]{2} + 4\sqrt[5]{2} - \frac{1}{11} + \frac{4}{6} - 4\right) =$$

$$= \frac{\pi}{4}\left(\frac{20-72+180}{33}\sqrt[5]{2} + \frac{-3+36-180}{33}\right) = \frac{\pi}{4}\left(\frac{128}{33}\sqrt[5]{2} - \frac{147}{33}\right)$$

$$\int_V x^2\,dx\,dy\,dz = \frac{\pi}{36} + \frac{\pi}{4}\left(\frac{128}{33}\sqrt[5]{2} - \frac{147}{33}\right) = \frac{\pi}{4}\left(\frac{128}{33}\sqrt[5]{2} - \frac{147}{33} + \frac{1}{9}\right)$$

$$= \frac{\pi}{4}\left(\frac{128}{33}\sqrt[5]{2} - \frac{144}{33}\right) = \pi\left(\frac{32}{33}\sqrt[5]{2} - \frac{4}{3}\right)$$

3. Sia $D := [0,+\infty[\times[0,+\infty[$.

(a) Provare che

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^4}\,dx\,dy < +\infty.$$

(b) Stabilire per quali $\alpha > 0$ converge

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy.$$

(a)

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^4}\,dx\,dy = \int_0^{\pi/2}\int_0^{+\infty} \frac{\log(1+\rho^2\sin\theta\cos\theta)}{1+\rho^8}\,\rho\,d\rho\,d\theta =$$

$$\le \int_0^{\pi/2}\int_0^{+\infty} \frac{\log(1+\rho^2)}{1+\rho^8}\,d\rho\,d\theta = \frac{\pi}{2}\int_0^{+\infty}\frac{\log(1+\rho^2)}{1+\rho^8}\,d\rho \le$$

NON INDEFINITO

CONVERGE

CONVERGE

$$\le \frac{\pi}{2}\int_0^1 \frac{\log(1+\rho^2)}{1+\rho^8}\,d\rho + \frac{\pi}{2}\int_1^{+\infty}\frac{\rho^2}{\rho^8}\,d\rho < +\infty \qquad \begin{cases}\log(1+x)\le x\\ 1+x > x\end{cases}$$

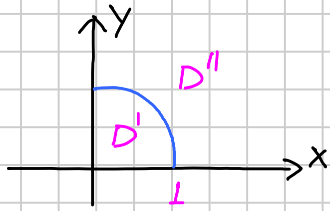
NON INDEFINITO

CONVERGE $\forall\alpha$

INDEFINITO

(b)

$$\int_D \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy = \int_{D'} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy + \int_{D''} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy$$

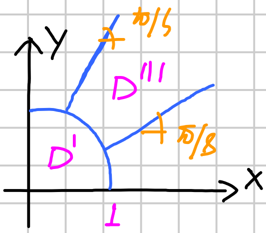


$$\int_{D''} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy = \int_0^{\pi/2}\int_1^{+\infty} \frac{\log(1+\frac{1}{2}\rho^2\sin 2\theta)}{1+\rho^{2\alpha}}\,\rho\,d\rho\,d\theta$$

$$\le \frac{\pi}{2}\int_1^{+\infty} \frac{\log(1+\frac{1}{2}\rho^2)\,\rho}{1+\rho^{2\alpha}}\,d\rho < +\infty \quad \text{PER } 2\alpha-1>1 \rightsquigarrow \alpha>1$$

INFATTI: $\rho\to+\infty \quad \dfrac{\log(1+\frac{1}{2}\rho^2)\,\rho}{1+\rho^{2\alpha}} \sim \dfrac{\rho}{\rho^{2\alpha}} = \dfrac{1}{\rho^{2\alpha-1}}$

PER $\alpha\le 1$



$$\int_{D''} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy \ge \int_{D'''} \frac{\log(1+xy)}{1+(x^2+y^2)^\alpha}\,dx\,dy =$$

$$= \int_{\pi/8}^{\pi/4}\int_1^{+\infty} \frac{\log(1+\frac{1}{2}\rho^2\sin 2\theta)}{1+\rho^{2\alpha}}\,\rho\,d\rho\,d\theta \ge \frac{\pi}{8}\int_1^{+\infty}\frac{\log(1+\frac{\sqrt{2}}{4}\rho^2)\,\rho\,d\rho\,d\theta}{1+\rho^{2\alpha}} = +\infty$$

$$\frac{\log(1+\frac{\sqrt{2}}{4}\rho^2)\,\rho}{1+\rho^{2\alpha}} \sim \frac{\rho}{\rho^{2\alpha}} = \frac{1}{\rho^{2\alpha-1}}$$

4. Sia D il dominio di $\mathbb{R}^2$ delimitato dalla curva γ parametrizzata da $\gamma(t) = (t - t^2, t^3)$ con $0 \le t \le 1$ e dall'asse delle y.

(a) Fare un disegno approssimativo di D.

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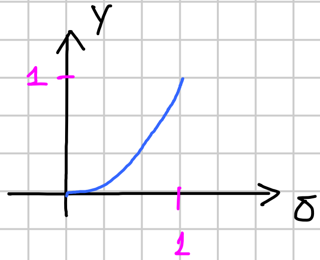
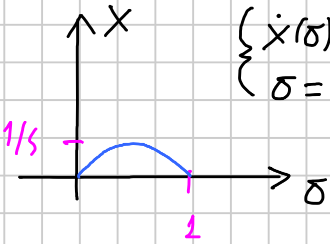
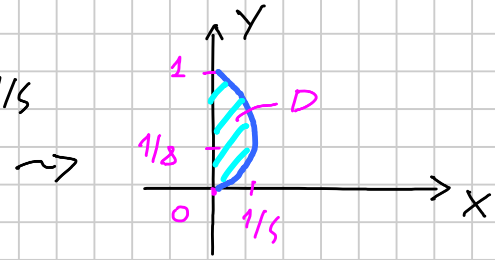
$$\int_D y^3 \, dx \, dy.$$

(a)





$$\begin{cases} \dot{x}(t) = 1 - 2t = 0 \\ t = 1/2 \quad x_{MAX} = 1/4 \end{cases}$$

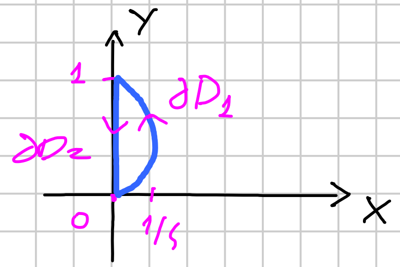


(b) $\displaystyle\int_D y^3 \, dx \, dy$

MODO 1 "TEOREMA DI GAUSS-GREEN"

$$y^3 = \operatorname{div} \vec{E} = A_x + B_y \qquad \vec{E} = (0, y^4/4)$$

$$\int_D y^3 \, dx \, dy = -\int_{\partial D} y^4/4 \, dx =$$



$$= -\int_{\partial D_1} y^4/4 \, dx - \underbrace{\cancel{\int_{\partial D_2} y^4/4 \, dx}}_{=0} =$$

$\partial D_2: \ 0 \le t \le 1 \quad \gamma_2(t) = (0, 1-t)$

$$= -\frac{1}{4}\int_0^1 t^{12} \cdot (1-2t) \, dt = -\frac{1}{4}\int_0^1 t^{12} - 2t^{13} \, dt =$$

$$= -\frac{1}{4}\left[\frac{t^{13}}{13} - \frac{t^{14}}{7}\right]_0^1 = -\frac{1}{4}\left(\frac{1}{13} - \frac{1}{7}\right) = -\frac{1}{4} \cdot \frac{7-13}{91} = \frac{3}{182}$$

MODO 2 "D COME INSIEME NORMALE"

$$\begin{cases} y = t^3 \\ x = t - t^2 = \sqrt[3]{y} - \sqrt[3]{y^2} \end{cases}$$

$$\int_D y^3 \, dx \, dy = \int_0^1 \int_0^{\sqrt[3]{y} - \sqrt[3]{y^2}} y^3 \, dx \, dy = \int_0^1 y^3 \left(y^{1/3} - y^{2/3}\right) dy =$$

$$= \int_0^1 \left(y^{10/3} - y^{11/3}\right) dy = \left[\frac{3}{13} y^{13/3} - \frac{3}{14} y^{14/3} \right]_0^1 =$$

$$= \frac{3}{13} - \frac{3}{14} = \frac{42 - 39}{13 \cdot 14} = \frac{3}{13 \cdot 14} = \frac{3}{182}$$