(b) $\begin{cases} \langle 1,1\rangle = 2-1\cdot 1 = 1 \quad \langle 1,x\rangle = 0-1\cdot 2=-2 \quad \langle 1,x^2\rangle = \int_{-1}^{1} x^2 dx - 4 = \frac{2}{3}-4 = -\frac{10}{3} \\ \langle x,x\rangle = \frac{2}{3}-4=-\frac{10}{3} \quad \langle x,x^2\rangle = 0-2\cdot 4=-8 \quad \langle x^2,x^2\rangle = \frac{2}{5}-4\cdot 4 = -\frac{78}{5} \end{cases}$

$\rightsquigarrow B = \begin{pmatrix} 1 & -2 & -10/3 \\ -2 & -10/3 & -8 \\ -10/3 & -8 & -78/5 \end{pmatrix}$

SYLVESTER (1,2,3): $DET(1)=1$

$DET\begin{pmatrix} 1 & -2 \\ -2 & -10/3 \end{pmatrix} = -\frac{10}{3}-4 = -\frac{22}{3}$

$\rightsquigarrow$ INDEFINITA

3. Spazio vettoriale: $\mathbb{R}_{\leq 3}[x]$ con base canonica $\{1, x, x^2, x^3\}$. Elementi generici: $p(x)$ e $q(x)$. Stesse espressioni del punto precedente.

1) $\int_{-1}^{1} p(x)q(x)\,dx$

(a) SÌ (Vd. P.TO 2)

(b) $\langle 1,1\rangle = \int_{-1}^{1} dx = [x]_{-1}^{1} = 2 \qquad \langle 1,x\rangle = \int_{-1}^{1} x\,dx = [x^2/2]_{-1}^{1} = 0$

$\langle 1,x^2\rangle = \int_{-1}^{1} x^2 dx = [x^3/3]_{-1}^{1} = 2/3 \qquad \langle x,x\rangle = \int_{-1}^{1} x^2 dx = 2/3$

$\langle x,x^2\rangle = \int_{-1}^{1} x^3 dx = [x^4/4]_{-1}^{1} = 0 \qquad \langle x^2,x^2\rangle = \int_{-1}^{1} x^4 dx = [x^5/5]_{-1}^{1} = 2/5$

$\langle 1,x^3\rangle = 0 \quad \langle x,x^3\rangle = 2/5 \quad \langle x^2,x^3\rangle = 0 \quad \langle x^3,x^3\rangle = \int_{-1}^{1} x^6 dx = 2/7$

B'

$\rightsquigarrow B = \begin{pmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{pmatrix}$

$DET(B) = \frac{2}{7}\frac{32}{135} + \frac{2}{5}\left(\frac{8}{45} - \frac{8}{25}\right) = \frac{256}{23625}$

SYLVESTER (1,2,3,4): $DET(2)=2 \quad DET\begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} = 4/3 \quad DET(B') = 32/135 \quad DET(B) > 0$

$\overset{P}{+}\overset{P}{+}\overset{P}{+}\overset{P}{+}+ \quad n_+ = 4 \quad n_- = 0 \quad n_0 = 0 \rightsquigarrow$ DEF. POSITIVA

(c) $v_1 = 1 \quad v_2 = x \quad v_3 = x^2 \quad v_4 = x^3$

$w_1 = v_1 = 1 \qquad w_2 = x \qquad w_3 = x^2 - \frac{1}{3} \qquad \langle w_3, w_3\rangle = \frac{8}{45}$

$$W_4 = X^3 - \frac{\langle V_4,W_1\rangle}{\langle W_1,W_1\rangle} W_1 - \frac{\langle V_4,W_2\rangle}{\langle W_2,W_2\rangle} W_2 - \frac{\langle V_4,W_3\rangle}{\langle W_3,W_3\rangle} W_3 =$$

$$= X^3 - \frac{0}{1}\,1 - \frac{2/5}{2/3} X - \frac{0}{8/45}\left(X^2 - 1/3\right) = -\frac{3}{5}X + X^3$$

$$\langle W_4,W_4\rangle = \begin{pmatrix} 0 & 0 & 0 & -\frac{6}{25}+\frac{2}{7}\end{pmatrix}\begin{pmatrix}0\\ -3/5\\ 0\\ 1\end{pmatrix} = \frac{-42+50}{175} = \frac{8}{175}$$

$$W_1^* = \frac{W_1}{\|W_1\|} = \frac{1}{\sqrt{2}} \quad W_2^* = \frac{W_2}{\|W_2\|} = \sqrt{\frac{3}{2}}\,X \quad W_3^* = -\frac{\sqrt{5}}{2\sqrt{2}} + \frac{3\sqrt{5}}{2\sqrt{2}}X^2 \quad W_4^* = \frac{W_4}{\|W_4\|} = \frac{5\sqrt{14}}{4}\left(-\frac{3}{5}X + X^3\right)$$

$$M = \begin{pmatrix} 1/\sqrt{2} & 0 & -\sqrt{5}/2\sqrt{2} & 0 \\ 0 & \sqrt{3/2} & 0 & -3\sqrt{14}/4 \\ 0 & 0 & 3\sqrt{5}/2\sqrt{2} & 0 \\ 0 & 0 & 0 & 5\sqrt{14}/4 \end{pmatrix} \qquad M^T B M = I$$

2) $\int_0^1 p(x)q(x)\,dx$

(a) SI (Vd. P.TO 2)

(b) $\langle 1,1\rangle = \int_0^1 dx = [x]_0^1 = 1 \qquad \langle 1,x\rangle = \int_0^1 x\,dx = [x^2/2]_0^1 = 1/2$

$\langle 1,x^2\rangle = \int_0^1 x^2\,dx = [x^3/3]_0^1 = 1/3 \qquad \langle x,x\rangle = \int_0^1 x^2\,dx = 1/3$

$\langle x,x^2\rangle = \int_0^1 x^3\,dx = [x^4/4]_0^1 = 1/4 \qquad \langle x^2,x^2\rangle = \int_0^1 x^4\,dx = [x^5/5]_0^1 = 1/5$

$\langle 1,x^3\rangle = 1/4 \quad \langle x,x^3\rangle = 1/5 \quad \langle x^2,x^3\rangle = 1/6 \quad \langle x^3,x^3\rangle = 1/7$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix} \qquad B'$$

SILV. (1,2,3,4): $DET(1) = 1 \quad DET\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3\end{pmatrix} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad DET(B') = 1/2160 \quad DET(B) > 0$

$\overset{}{+}\overset{P}{\ }+\overset{P}{\ }+\overset{P}{\ }+\overset{P}{\ }+$ $\quad n_+ = 4 \quad n_- = 0 \quad n_0 = 0 \rightsquigarrow$ DEF. POSITIVA

(c) $v_1 = 1 \quad v_2 = x \quad v_3 = x^2 \quad v_5 = x^3$

$$w_1 = v_1 = 1 \quad w_2 = -\frac{1}{2} + x \quad w_3 = \frac{1}{6} - x + x^2$$

$$\langle w_2, w_2 \rangle = \frac{1}{12} \qquad \langle w_3, w_3 \rangle = \frac{1}{180}$$

$$\langle w_2, v_5 \rangle = \left(* \quad * \quad * \quad -\frac{1}{8} + \frac{1}{5} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{-5+8}{50} = \frac{3}{50}$$

$$\langle w_3, v_5 \rangle = \left(* \quad * \quad * \quad \frac{1}{25} - \frac{1}{5} + \frac{1}{6} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{5-25+20}{120} = \frac{1}{120}$$

$$\hat{w}_5 = x^3 - \frac{\langle v_5, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_5, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \frac{\langle v_5, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3 =$$

$$= x^3 - \frac{1/5}{1} 1 - \frac{3/50}{1/12} (-1/2 + x) - \frac{1/120}{1/180} (1/6 - x + x^2) =$$

$$= x^3 - \frac{1}{5} + \frac{9}{20} - \frac{9}{10} x - \frac{1}{5} + \frac{3}{2} x - \frac{3}{2} x^2 = -\frac{1}{20} + \frac{3}{5} x - \frac{3}{2} x^2 + x^3$$

$$w_5 = -1 + 12x - 30x^2 + 20x^3$$

$$\langle w_5, w_5 \rangle = \left(\cancel{-1} + \cancel{6} - \cancel{10} + \cancel{5} \quad -\cancel{1/2} + \cancel{5} - \cancel{\tfrac{15}{2}} + \cancel{5} \quad -\cancel{1/3} + \cancel{3} - \cancel{6} + \cancel{\tfrac{10}{3}} \quad -1/5 + \frac{12}{5} - 5 + \frac{20}{7} \right) \begin{pmatrix} -1 \\ 12 \\ -30 \\ 20 \end{pmatrix}$$

$$= -5 + 58 - 100 + 500/7 = \frac{-399+500}{7} = \frac{1}{7}$$

$$w_1^* = \frac{w_1}{\|w_1\|} = 1 \quad w_2^* = \frac{w_2}{\|w_2\|} = -\sqrt{3} + 2\sqrt{3}x \quad w_3^* = \frac{w_3}{\|w_3\|} = 6\sqrt{5}\left(\frac{1}{6} - x + x^2\right)$$

$$w_5^* = \frac{w_5}{\|w_5\|} = \sqrt{7}(-1 + 12x - 30x^2 + 20x^3)$$

$$M = \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & 12\sqrt{7} \\ 0 & 0 & 6\sqrt{5} & -30\sqrt{7} \\ 0 & 0 & 0 & 20\sqrt{7} \end{pmatrix} \qquad M^T B M = I$$

3) $\int_{-1}^{1} p'(x) q'(x)\, dx$

(a) SÍ (Vd. p.to 2)

(b) $<1,1> = \int_{-1}^{1} 0\, dx = 0 \quad <1,x> = \int_{-1}^{1} 0\, dx = 0 \quad <1,x^2> = \int_{-1}^{1} 0\, dx = 0$

$<x,x> = \int_{-1}^{1} 1\, dx = [x]_{-1}^{1} = 2 \quad <x,x^2> = \int_{-1}^{1} 2x\, dx = [x^2]_{-1}^{1} = 0$

$<x^2,x^2> = \int_{-1}^{1} 4x^2\, dx = [4x^3/3]_{-1}^{1} = 8/3$

$<1,x^3> = 0 \quad <x,x^3> = \int_{-1}^{1} 3x^2\, dx = [x^3]_{-1}^{1} = 2 \quad <x^2,x^3> = 0 \quad <x^3,x^3> = \int_{-1}^{1} 9x^4\, dx = 6$

$$\rightsquigarrow B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 8/3 & 0 \\ 0 & 2 & 0 & 6 \end{pmatrix}$$

$|B-\lambda I| = -\lambda\left[(2-\lambda)(\frac{8}{3}-\lambda)(6-\lambda) - 4(8/3-\lambda)\right] =$
$= -\lambda(\frac{8}{3}-\lambda)\left[\lambda^2 - 8\lambda + 12 - 4\right] = \lambda^4 - 8\lambda^3 + 8\lambda^2 - \frac{8}{3}\lambda^3 +$
$+ \frac{64}{3}\lambda^2 - \frac{64}{3}\lambda = \lambda^4 - \frac{32}{3}\lambda^3 + \frac{88}{3}\lambda^2 - \frac{64}{3}\lambda$

$n_+ = 3 \quad n_0 = 1 \quad n_- = 0 \rightsquigarrow$ SEMIDEF. POSITIVA

4) $\int_{0}^{1} p(x) q'(x)\, dx$

(a) NO (Vd. p.to 2)

5) $p(x)q(x)$ NON È $f: V \times V \to \mathbb{R}$

6) $p(0)q(0)$

(a) SÍ (Vd. p.to 2)

(b) $<1,1> = 1 \cdot 1 = 1 \quad <1,x> = 1 \cdot 0 = 0 \quad <1,x^2> = 1 \cdot 0 = 0$

$<x,x> = 0 \cdot 0 = 0 \quad <x,x^2> = 0 \cdot 0 = 0 \quad <x^2,x^2> = 0 \cdot 0 = 0$

$<1,x^3> = 0 \cdot 0 = 0 \quad <x,x^3> = 0 \cdot 0 = 0 \quad <x^2,x^3> = 0 \cdot 0 = 0 \quad <x^3,x^3> = 0 \cdot 0 = 0$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$n_+ = 1 \quad n_0 = 3 \quad n_- = 0 \rightsquigarrow$ SEMIDEF. POSITIVA

7) $p(1)q(0)$

(a) NO (Vd. p.to 2)

8) $p(0)q(1)+p(1)q(0)$

(a) SÍ (Vd. p.to 2)

(b) $\langle 1,1\rangle = 1+1=2$ $\langle 1,x\rangle = 1+0=1$ $\langle 1,x^2\rangle = 1+0=1$

$\langle x,x\rangle = 0+0=0$ $\langle x,x^2\rangle = 0+0=0$ $\langle x^2,x^2\rangle = 0+0=0$

$\langle 1,x^3\rangle = 1+0=1$ $\langle x,x^3\rangle = 0+0=0$ $\langle x^2,x^3\rangle = 0+0=0$ $\langle x^3,x^3\rangle = 0+0=0$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

COMPL. QUAD.: $x^2+2xy+2xz+2xw =$

$=(x+y+z+w)^2-y^2-z^2-w^2-2yz-2yw-2zw =$

$=(x+y+z+w)^2-(y+z+w)^2$

$n_+=1$ $n_-=1$ $n_0=2$ $\rightsquigarrow$ INDEFINITA

9) $p'(0)q'(0)$

(a) SÍ (Vd. p.to 2)

(b) $\begin{cases} \langle 1,1\rangle = 0 \quad \langle 1,x\rangle = 0 \quad \langle 1,x^2\rangle = 0 \quad \langle 1,x^3\rangle = 0 \quad \langle x,x^3\rangle = 0 \\ \langle x,x\rangle = 1 \quad \langle x,x^2\rangle = 0 \quad \langle x^2,x^2\rangle = 0 \quad \langle x^2,x^3\rangle = 0 \quad \langle x^3,x^3\rangle = 0 \end{cases}$

$$\rightsquigarrow B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$n_+=1$ $n_-=0$ $n_0=3$

$\rightsquigarrow$ SEMIDEF. POSITIVA

10) $p(0)q(0)+p(1)q(1)+p(-1)q(-1)$

(a) SÍ (Vd. p.to 2)

(b) $\langle 1,1\rangle = 1+1+1=3$ $\langle 1,x\rangle = 0+1-1=0$ $\langle 1,x^2\rangle = 0+1+1=2$

$\langle x,x\rangle = 0+1+1=2$ $\langle x,x^2\rangle = 0+1-1=0$ $\langle x^2,x^2\rangle = 0+1+1=2$

$\langle 1,x^3\rangle = 0+1-1=0$ $\langle x,x^3\rangle = 0+1+1=2$ $\langle x^2,x^3\rangle = 0+1-1=0$ $\langle x^3,x^3\rangle = 0+1+1=2$

$$\rightsquigarrow B = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

B'

$\det(B') = 4$

$\det(B) = 2\cdot 4 + 2(8-12) = 0$

$$|B-\lambda I| = \begin{vmatrix} 3-\lambda & 0 & 2 & 0 \\ 0 & 2-\lambda & 0 & 2 \\ 2 & 0 & 2-\lambda & 0 \\ 0 & 2 & 0 & 2-\lambda \end{vmatrix} = \begin{cases} (3-\lambda)\left[(2-\lambda)^3-4(2-\lambda)\right]+2\left[8-2(2-\lambda)^2\right] = \\ =(3-\lambda)(2-\lambda)\left[\lambda^2-4\lambda\right]+2\left[-2\lambda^2+8\lambda\right] = \\ =(\lambda^2-4\lambda)\left[\lambda^2-5\lambda+6-4\right]=(\lambda^2-4\lambda)(\lambda^2-5\lambda+2) = \\ =\lambda^4-5\lambda^3+2\lambda^2-4\lambda^3+20\lambda^2-8\lambda = \lambda^4-9\lambda^3+22\lambda^2-8\lambda \end{cases}$$

CARTESIO: $m_+ = 3$ $m_0 = 1$ $m_- = 4-3-1 = 0$ $\rightsquigarrow$ SEMIDEF. POSITIVA

11) $p(0)q(0)+p'(1)q'(1)+p(-1)q(-1)$

(a) SI (Vd. p.to 2)

(b) $\langle 1,1\rangle = 1+0+1=2$ $\langle 1,x\rangle = 0+0-1=-1$ $\langle 1,x^2\rangle = 0+0+1=1$

$\langle x,x\rangle = 0+1+1=2$ $\langle x,x^2\rangle = 0+2-1=1$ $\langle x^2,x^2\rangle = 0+4+1=5$

$\langle 1,x^3\rangle = 0+0-1=-1$ $\langle x,x^3\rangle = 0+3+1=4$ $\langle x^2,x^3\rangle = 0+6-1=5$ $\langle x^3,x^3\rangle = 0+9+1=10$

$$\rightsquigarrow B = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 5 & 5 \\ -1 & 4 & 5 & 10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -2 & 2 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & -3 & -9 & -11 \\ 0 & 5 & 10 & 15 \end{pmatrix}$$

B'

$2\,DET(B) = 2(-403-330+$
$-270+403+330+270) =$
$=0$

$$2x^2+2y^2+5z^2+10w^2-2xy+2xz-2xw+2yz+8yw+10zw =$$

$$=2(x-y/2+z/2-w/2)^2+\frac{3}{2}y^2+\frac{9}{2}z^2+\frac{19}{2}w^2+3yz+7yw+11zw =$$

$$=2(x-y/2+z/2-w/2)^2+\frac{3}{2}\left(y+z+\frac{7}{3}w\right)^2+3z^2+\frac{4}{3}w^2+4zw$$

$$=2(x-y/2+z/2-w/2)^2+\frac{3}{2}\left(y+z+\frac{7}{3}w\right)^2+3\left(z+\frac{2}{3}w\right)^2$$

$m_+=3$ $m_0=1$ $m_-=0$ $\rightsquigarrow$ SEMI DEF. POSITIVA

12) $(p(0)+2p(1))(q(0)+2q(1))$

(a) SI (Vd. p.to 2)

(b) $\langle 1,1\rangle = 3\cdot 3 = 9$ $\langle 1,x\rangle = 3\cdot 2 = 6$ $\langle 1,x^2\rangle = 3\cdot 2 = 6$

$\langle x,x\rangle = 2\cdot 2 = 4$ $\langle x,x^2\rangle = 2\cdot 2 = 4$ $\langle x^2,x^2\rangle = 2\cdot 2 = 4$

$\langle 1,x^3\rangle = 3\cdot 2 = 6$ $\langle x,x^3\rangle = 2\cdot 2 = 4$ $\langle x^2,x^3\rangle = 4$ $\langle x^3,x^3\rangle = 4$

$$\rightsquigarrow B = \begin{pmatrix} 9 & 6 & 6 & 6 \\ 6 & 5 & 5 & 5 \\ 6 & 5 & 5 & 5 \\ 6 & 5 & 5 & 5 \end{pmatrix} \rightsquigarrow \text{RANGO } 1 \quad n_0 = 2$$

C.Q.: $9x^2+5y^2+5z^2+5w^2+12xy+12xz+12xw-18yz+8yw+8zw =$

$= (3x+2y+2z+2w)^2 \quad n_+ = 1 \quad n_0 = 3 \quad n_- = 0 \rightsquigarrow$ SEMIDEF. POSITIVA

13) $(p(0)+2q(1))(q(0)+2p(1))$

(a) NO (Vd. p.to 2)

15) $p(0)q(0)-p(1)q(1)$

(a) SI' (Vd. p.to 2)

(b) $\langle 1,1 \rangle = 1-1 = 0 \quad \langle 1,x \rangle = 0-1 = -1 \quad \langle 1,x^2 \rangle = 0-1 = -1$

$\langle x,x \rangle = 0-1 = -1 \quad \langle x,x^2 \rangle = 0-1 = -1 \quad \langle x^2,x^2 \rangle = 0-1 = -1$

$\langle 1,x^3 \rangle = 0-1 = -1 \quad \langle x,x^3 \rangle = 0-1 = -1 \quad \langle x^2,x^3 \rangle = 0-1 = -1 \quad \langle x^3,x^3 \rangle = 0-1 = -1$

$$\rightsquigarrow B = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \rightsquigarrow \text{RANGO} = 2 \quad n_0 = 2$$

C.Q.: $-y^2-z^2-w^2-2xy-2xz-2xw-2yz-2yw-2zw =$

$= -(x+y+z+w)+x^2 \quad n_+ = 1 \quad n_- = 1 \quad n_0 = 2 \rightsquigarrow$ INDEFINITA

13) $\int_{-1}^{1} p'(x)q'(x)\,dx + p(3)q(3)$

(a) SI' (Vd. p.to 2)

(b) $\langle 1,1 \rangle = 0+1\cdot 1 = 1 \quad \langle 1,x \rangle = 0+1\cdot 3 = 3 \quad \langle 1,x^2 \rangle = 0+1\cdot 9 = 9$

$\langle x,x \rangle = \int_{-1}^{1} dx + 3\cdot 3 = 2+9 = 11 \quad \langle x,x^2 \rangle = \int_{-1}^{1} 2x\,dx + 3\cdot 9 = 0+27 = 27$

$\langle x^2,x^2 \rangle = \int_{-1}^{1} 4x^2\,dx + 9\cdot 9 = [4/3\,x^3]_{-1}^{1} + 81 = \frac{8}{3}+81 = \frac{251}{3}$

$\langle 1,x^3 \rangle = 0+1\cdot 27 = 27 \quad \langle x,x^3 \rangle = \int_{-1}^{1} 3x^2\,dx + 3\cdot 27 = 83$

$$\langle x^2, x^3\rangle = \int_{-1}^{1} 6x^3\,dx + 9\cdot 27 = 243$$

$$\langle x^3, x^3\rangle = \int_{-1}^{1} 9x^4\,dx + 27\cdot 27 = \left[\tfrac{9}{5}x^5\right]_{-1}^{1} + 729 = \frac{18}{5} + 729 = \frac{3663}{5}$$

$$\rightsquigarrow B = \left(\begin{array}{ccc|c} 1 & 3 & 9 & 27 \\ 3 & 11 & 27 & 83 \\ 9 & 27 & 251/3 & 243 \\ \hline 27 & 83 & 243 & 3663/5 \end{array}\right) \qquad B'$$

$DET(B') = 16/3$

$DET(B) = 128/15$

SYLVESTER(1,2,3,4): $DET(1) = 1 \quad DET\begin{pmatrix} 1 & 3 \\ 3 & 11 \end{pmatrix} = 2 \quad DET(B') = 16/3 \quad DET(B) > 0$

$\overset{P}{+}\overset{P}{+}\overset{P}{+}\overset{P}{+}+ \quad n_+ = 4 \quad n_- = 0 \quad n_0 = 0 \rightsquigarrow$ DEF. POSITIVA

(c) $V_1 = 1 \quad V_2 = x \quad V_3 = x^2 \quad V_4 = x^3$

$W_1 = V_1 \quad W_2 = -3 + x \quad W_3 = -9 + x^2 \quad \langle W_2, W_2\rangle = 2 \quad \langle W_3, W_3\rangle = \frac{8}{3}$

$$\langle V_4, W_2\rangle = (27 \ \ 83 \ \ 243 \ \ 3663/5)\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -81 + 83 = 2$$

$$\langle V_4, W_3\rangle = (27 \ \ 83 \ \ 243 \ \ 3663/5)\begin{pmatrix} -9 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -243 + 243 = 0$$

$$W_4 = x^3 - \frac{\langle V_4, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle V_4, W_2\rangle}{\langle W_2, W_2\rangle} W_2 - \frac{\langle V_4, W_3\rangle}{\langle W_3, W_3\rangle} W_3 =$$

$$= x^3 - \frac{27}{1}\,1 - \frac{2}{2}(-3+x) - \frac{0}{8/3}(-9+x^2) = x^3 - 27 + 3 - x = -24 - x + x^3$$

$$\langle W_4, W_4\rangle = \left(\overset{0}{\cancel{-24}\ \cancel{-3}+\cancel{27}} \quad \overset{0}{\cancel{-72}-\cancel{11}+\cancel{83}} \quad \overset{0}{-2\cancel{16}-\cancel{27}+\cancel{243}} \quad -648-83+\tfrac{3663}{5}\right)\begin{pmatrix} -24 \\ -1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \frac{-3240 - 415 + 3663}{5} = \frac{8}{5}$$

$$W_1^* = \frac{W_1}{\|W_1\|} = 1 \quad W_2^* = \frac{W_2}{\|W_2\|} = \frac{1}{\sqrt{2}}(-3+x) \quad W_3^* = \frac{W_3}{\|W_3\|} = \frac{\sqrt{3}}{2\sqrt{2}}(-9+x^2)$$

$$W_4^* = \frac{W_4}{\|W_4\|} = \frac{\sqrt{5}}{2\sqrt{2}}(-24 - x + x^3)$$

$$M = \begin{pmatrix} 1 & -3/\sqrt{2} & -9\sqrt{3}/2\sqrt{2} & -12\sqrt{5}/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -\sqrt{5}/2\sqrt{2} \\ 0 & 0 & \sqrt{3}/2\sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{5}/2\sqrt{2} \end{pmatrix} \qquad M^t B M = I$$

16) $\int_{-2}^{2} p(x)q(x)dx - p(2)q(2)$

(a) SI (Vd. P.TO 2)

(b) $\langle 1,1\rangle = 2-1\cdot 1 = 1$ $\quad \langle 1,x\rangle = 0-1\cdot 2 = -2$ $\quad \langle 1,x^2\rangle = \int_{-2}^{2} x^2 dx - 4 = \frac{2}{3} - 4 = -\frac{10}{3}$

$\langle x,x\rangle = \frac{2}{3} - 4 = -\frac{10}{3}$ $\quad \langle x,x^2\rangle = 0 - 2\cdot 4 = -8$ $\quad \langle x^2,x^2\rangle = \frac{2}{5} - 4\cdot 4 = -\frac{78}{5}$

$\langle 1,x^3\rangle = 0 - 1\cdot 8 = -8$ $\quad \langle x,x^3\rangle = \frac{2}{5} - 16 = \frac{78}{5}$ $\quad \langle x^2,x^3\rangle = 0 - 32 = -32$

$\langle x^3,x^3\rangle = \int_{-2}^{2} x^6 dx - 8\cdot 8 = \left[x^7/7\right]_{-2}^{2} - 64 = 2/7 - 64 = -\frac{556}{7}$

$$\leadsto B = \begin{pmatrix} 1 & -2 & -10/3 & -8 \\ -2 & -10/3 & -8 & 78/5 \\ -10/3 & -8 & -78/5 & -32 \\ -8 & 78/5 & -32 & -556/7 \end{pmatrix}$$

SYLVESTER (1,2,3,4): DET(1) = 1

$\mathrm{DET}\begin{pmatrix} 1 & -2 \\ -2 & -10/3 \end{pmatrix} = -\frac{10}{3} - 4 = -\frac{22}{3}$

$\leadsto$ INDEFINITA

4. Spazio vettoriale: $M_{2\times 2}$. Elementi generici: A e B. Espressioni:

1) AB, 2) $AB + BA$, 3) $\mathrm{Tr}(AB)$, 4) $\mathrm{Tr}(A)\cdot\mathrm{Tr}(B)$, 5) $\mathrm{Tr}(A^tB)$, 6) $\mathrm{Tr}(AB^t)$,

7) $\det(AB)$, 8) $(1,0)AB\begin{pmatrix}0\\1\end{pmatrix}$, 9) $(1,0)(AB+BA)\begin{pmatrix}0\\1\end{pmatrix}$, 10) $\det A\cdot\det B$.

1) $AB \to$ NON È $f: V\times V \to \mathbb{R}$

2) $AB + BA \to$ NON È $f: V\times V \to \mathbb{R}$

3) $\mathrm{Tr}(AB)$

$$\overset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \overset{B}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} \leadsto \mathrm{Tr}(AB) = ae + bg + cf + dh$$

$(\leadsto x^2 + 2yz + w^2)$

(a) (i) SIMM. $Tr(AB) = ae + bg + cf + dh = Tr(BA) = ea + fc + gb + hd$ sí

(ii) LIN. $Tr([A + \hat{A}]B) = [ae + bg + cf + dh] + [\hat{a}e + \hat{b}g + \hat{c}f + \hat{d}h]$ sí

(iii) LIN. $Tr(\lambda AB) = \lambda[ae + bg + cf + dh]$ sí

(b) $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{cases} \langle e_1, e_1 \rangle = Tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \quad \langle e_1, e_2 \rangle = Tr\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_1, e_3 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \\ \langle e_1, e_4 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_2, e_2 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_2, e_3 \rangle = Tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \\ \langle e_2, e_4 \rangle = Tr\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_3, e_3 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_3, e_4 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_4, e_4 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \end{cases}$$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c.a.: $x^2 + w^2 + 2yz = x^2 + w^2 + \frac{1}{2}(y+z)^2 - \frac{1}{2}(y-z)^2$

$n_+ = 3$ $n_- = 1$ $n_0 = 0$ $\rightsquigarrow$ INDEFINITA

5) $Tr(A) \cdot Tr(B)$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ $\quad Tr(A) \cdot Tr(B) = (a+d)(e+h)$

$(\rightsquigarrow (x+w) \cdot (x+w))$

(a) (i) SIMM. $Tr(B) \cdot Tr(A) = Tr(A) \cdot Tr(B)$ sí

(ii) LIN. $Tr(A + \hat{A}) \cdot Tr(B) = (a+d)(e+h) + (\hat{a} + \hat{b})(e+h)$ sí

(iii) LIN. $Tr(\lambda A)\, Tr(B) = (\lambda a + \lambda d)(e+h) = \lambda(Tr(A) \cdot Tr(B))$ sí

(b) $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{cases} \langle e_1, e_1 \rangle = 1 \cdot 1 = 1 \quad \langle e_1, e_2 \rangle = 1 \cdot 0 = 0 \quad \langle e_1, e_3 \rangle = 1 \cdot 0 = 0 \quad \langle e_1, e_4 \rangle = 1 \cdot 1 = 1 \\ \langle e_2, e_2 \rangle = 0 \cdot 0 = 0 \quad \langle e_2, e_3 \rangle = 0 \cdot 0 = 0 \quad \langle e_2, e_4 \rangle = 0 \cdot 1 = 0 \quad \langle e_3, e_3 \rangle = 0 \cdot 0 = 0 \\ \langle e_3, e_4 \rangle = 0 \cdot 1 = 0 \quad \langle e_4, e_4 \rangle = 1 \cdot 1 = 1 \end{cases}$$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

c.a.: $x^2 + w^2 + 2xw = (x+w)^2$

$n_+ = 1$ $n_0 = 3$ $n_- = 0$ $\rightsquigarrow$ SEMIDEF. POSITIVA

5) $Tr(A^{\delta}B)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \qquad Tr(A^{\delta}B) = ae + cg + bf + dh$$

$(\rightsquigarrow x^2+y^2+z^2+w^2)$

(a) (i) SIMM. $Tr(B^{\delta}A) = ea + gc + fb + hd = Tr(A^{\delta}B)$ SÍ

(ii) LIN. $Tr[(A+\hat{A})^{\delta}B] = (a+\hat{a})e + (c+\hat{c})g + (b+\hat{b})f + (d+\hat{d})h =$

$= (ae + cg + bf + dh) + (\hat{a}e + \hat{c}g + \hat{b}f + \hat{d}h)$ SÍ

(iii) LIN. $Tr(\lambda A^{\delta}B) = \lambda ae + \lambda cg + \lambda bf + \lambda dh = \lambda Tr(A^{\delta}B)$ SÍ

(b) $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{cases} \langle e_1, e_1 \rangle = Tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \quad \langle e_1, e_2 \rangle = Tr\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_1, e_3 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \\ \langle e_1, e_4 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_2, e_2 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \quad \langle e_2, e_3 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \\ \langle e_2, e_4 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_3, e_3 \rangle = Tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \quad \langle e_3, e_4 \rangle = Tr\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \quad \langle e_4, e_4 \rangle = Tr\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 \end{cases}$$

$$\rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$n_+ = 4 \quad n_- = 0 \quad n_0 = 0 \rightsquigarrow$ DEF. POSITIVA

$(\rightsquigarrow x^2+y^2+z^2+w^2)$

(c) $w_1^* = e_1 \quad w_2^* = e_2 \quad w_3^* = e_3 \quad w_4^* = e_4$

6) $Tr(AB^{\delta})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \qquad Tr(AB^{\delta}) = ae + bf + cg + dh$$

$(\rightsquigarrow x^2+y^2+z^2+w^2)$

(a) (i) SIMM. $Tr(BA^{\delta}) = ea + fb + gc + hd = Tr(AB^{\delta})$ SÍ

(ii) LIN. $Tr[(A+\hat{A})B^{\delta}] = (a+\hat{a})e + (b+\hat{b})f + (c+\hat{c})g + (d+\hat{d})h =$

$= (ae + bf + cg + dh) + (\hat{a}e + \hat{b}f + \hat{c}g + \hat{d}h)$ SÍ

(iii) LIN. $Tr(\lambda AB^{\delta}) = \lambda ae + \lambda bf + \lambda cg + \lambda dh = \lambda Tr(AB^{\delta})$ SÍ

(b) $e_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $e_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $e_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $e_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$$\begin{cases} \langle e_1,e_1\rangle = Tr\begin{pmatrix}1&0\\0&0\end{pmatrix}=1 \quad \langle e_1,e_2\rangle = Tr\begin{pmatrix}0&0\\0&0\end{pmatrix}=0 \quad \langle e_1,e_3\rangle = Tr\begin{pmatrix}0&1\\0&0\end{pmatrix}=0 \\ \langle e_1,e_4\rangle = Tr\begin{pmatrix}0&0\\0&0\end{pmatrix}=0 \quad \langle e_2,e_2\rangle = Tr\begin{pmatrix}1&0\\0&0\end{pmatrix}=1 \quad \langle e_2,e_3\rangle = Tr\begin{pmatrix}0&0\\0&0\end{pmatrix}=0 \\ \langle e_2,e_4\rangle = Tr\begin{pmatrix}0&0\\0&0\end{pmatrix}=0 \quad \langle e_3,e_3\rangle = Tr\begin{pmatrix}0&0\\0&1\end{pmatrix}=1 \quad \langle e_3,e_4\rangle = Tr\begin{pmatrix}0&0\\0&0\end{pmatrix}=0 \quad \langle e_4,e_4\rangle = Tr\begin{pmatrix}0&0\\0&1\end{pmatrix}=1 \end{cases}$$

$$\rightsquigarrow B=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$n_+=4$ $n_-=0$ $n_0=0$ $\rightsquigarrow$ DEF. POSITIVA

$(\rightsquigarrow x^2+y^2+z^2+w^2)$

(c) $w_1^*=e_1$ $w_2^*=e_2$ $w_3^*=e_3$ $w_4^*=e_4$

7) $DET(AB)$

T. DI BINET

(a) (i) SIMM. $DET(BA)=DET(B)\cdot DET(A)=DET(AB)$ SÍ

(ii) LIN. $DET[(A+\hat{A})B]=DET(AB+\hat{A}B)\neq DET(AB)+DET(\hat{A}B)$ NO

(iii) LIN. $DET(\lambda A\cdot B)=DET(\lambda A)\cdot DET(B)=\lambda^2 DET(AB)$ NO

8) $(1,0)\,AB\begin{pmatrix}0\\1\end{pmatrix}$

$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $B=\begin{pmatrix}e&f\\g&h\end{pmatrix}$ $(1,0)\,AB\begin{pmatrix}0\\1\end{pmatrix}=(a\ b)\begin{pmatrix}f\\h\end{pmatrix}=af+bh$

(a) (i) $(1,0)\,BA\begin{pmatrix}0\\1\end{pmatrix}=(e\ f)\begin{pmatrix}b\\d\end{pmatrix}=eb+fd\neq(1,0)\,AB\begin{pmatrix}0\\1\end{pmatrix}$ NO

(ii) $(1,0)(A+\hat{A})B\begin{pmatrix}0\\1\end{pmatrix}=(a+\hat{a}\ \ b+\hat{b})\begin{pmatrix}f\\h\end{pmatrix}=(af+bh)+(\hat{a}f+\hat{b}h)$ SÍ

(iii) $(1,0)\lambda AB\begin{pmatrix}0\\1\end{pmatrix}=(\lambda a\ \ \lambda b)\begin{pmatrix}f\\h\end{pmatrix}=\lambda[(1,0)\,AB\begin{pmatrix}0\\1\end{pmatrix}]$ SÍ

9) $(1\ 0)(AB+BA)\begin{pmatrix}0\\1\end{pmatrix}=(1,0)\,AB\begin{pmatrix}0\\1\end{pmatrix}+(1,0)\,BA\begin{pmatrix}0\\1\end{pmatrix}$

$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $B=\begin{pmatrix}e&f\\g&h\end{pmatrix}$ $(1\ 0)(AB+BA)\begin{pmatrix}0\\1\end{pmatrix}=af+bh+eb+fd$

$(\rightsquigarrow 2xy+2yw)$

(a) (i) SIMM. $(1\ 0)(AB+BA)\begin{pmatrix}0\\1\end{pmatrix}=(1\ 0)(BA+AB)\begin{pmatrix}0\\1\end{pmatrix}$ SÍ

(ii), (iii) SÍ

(b) $e_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $e_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $e_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $e_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$$\begin{cases} \langle e_1,e_1\rangle=0 \quad \langle e_1,e_2\rangle=1 \quad \langle e_1,e_3\rangle=0 \quad \langle e_1,e_4\rangle=0 \\ \langle e_2,e_2\rangle=0 \quad \langle e_2,e_3\rangle=0 \quad \langle e_2,e_4\rangle=1 \quad \langle e_3,e_3\rangle=0 \\ \langle e_3,e_4\rangle=0 \quad \langle e_4,e_4\rangle=0 \end{cases}$$

$$\rightsquigarrow B=\begin{pmatrix}0&1&0&0\\1&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}$$

C.Q.: $2xy+2yw=$

$=\frac{1}{2}(x+y)^2-\frac{1}{2}(x-y)^2+\frac{1}{2}(y+w)^2-\frac{1}{2}(y-w)^2$

$n_+=2$ $n_-=2$ $n_0=0$ $\rightsquigarrow$ INDEFINITA

10) DET(A)·DET(B)

(a) DET(A)·DET(B) = DET(A·B) $\rightsquigarrow$ V.d.C. P.TO "7)"

(i) SIMM. SÍ

(ii), (iii) LIN. NO