Forme canoniche 2

Argomenti: forme canoniche di applicazioni lineari **Difficoltà:** ⋆⋆⋆

Prerequisiti: autovalori, autovettori, forme canoniche, matrici di cambio di base

---

1. Determinare, per ciascuna delle seguenti matrici, la forma canonica reale e quella complessa (se diversa da quella reale). Determinare anche, nel caso reale ed eventualmente nel caso complesso, una possibile matrice di cambio di base che realizza il passaggio alla forma canonica.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 8 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 2 \\ -8 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

2. Determinare per quali valori del parametro reale a la matrice $\begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$

   (a) è diagonalizzabile sui reali,

   (b) è diagonalizzabile sui reali mediante una matrice ortogonale,

   (c) ha il ker non banale,

   (d) ammette l'autovalore $\lambda = 1$,

   (e) ammette l'autovalore $\lambda = 2 + i$,

   (f) ammette $(1, 5)$ come autovettore,

   (g) non è diagonalizzabile sui complessi.

3. Determinare per quali valori del parametri reali a e b la matrice $\begin{pmatrix} a & 1 \\ b & 2 \end{pmatrix}$

   (a) ammette autovalori immaginari puri,

   (b) ammette l'autovalore $\lambda = 3 + 5i$,

   (c) è diagonalizzabile sui reali mediante una matrice ortogonale e rappresenta un'applicazione lineare non surgettiva,

   (d) ammette $\lambda = 7$ come autovalore con autovettore corrispondente $(1, 1)$,

   (e) ammette $\lambda = -1$ come autovalore ma non è diagonalizzabile sui complessi,

   (f) è simile alla matrice $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

SOLUZIONI REV.02
BY G IMUSI 08/01/15

1. Determinare, per ciascuna delle seguenti matrici, la forma canonica reale e quella complessa (se diversa da quella reale). Determinare anche, nel caso reale ed eventualmente nel caso complesso, una possibile matrice di cambio di base che realizza il passaggio alla forma canonica.

$$(a)\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad (b)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad (c)\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad (d)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad (e)\begin{pmatrix}0&1\\4&0\end{pmatrix}$$

$$(f)\begin{pmatrix}0&-1\\4&0\end{pmatrix}\quad (g)\begin{pmatrix}1&-3\\-1&3\end{pmatrix}\quad (h)\begin{pmatrix}1&1\\-1&3\end{pmatrix}\quad (i)\begin{pmatrix}1&1\\0&3\end{pmatrix}\quad (l)\begin{pmatrix}1&1\\-1&-1\end{pmatrix}$$

$$(m)\begin{pmatrix}1&0\\-1&-1\end{pmatrix}\quad (n)\begin{pmatrix}1&2\\4&-1\end{pmatrix}\quad (o)\begin{pmatrix}1&-2\\5&-1\end{pmatrix}\quad (p)\begin{pmatrix}3&0\\3&3\end{pmatrix}\quad (q)\begin{pmatrix}3&3\\3&3\end{pmatrix}$$

$$(r)\begin{pmatrix}-1&2\\8&-1\end{pmatrix}\quad (s)\begin{pmatrix}-1&2\\-8&-1\end{pmatrix}\quad (t)\begin{pmatrix}0&3\\1&0\end{pmatrix}\quad (u)\begin{pmatrix}1&2\\3&4\end{pmatrix}\quad (v)\begin{pmatrix}1&2\\-3&4\end{pmatrix}$$

$$(a)\quad \begin{pmatrix}1&0\\0&1\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad M=I \qquad\qquad (b)\quad \begin{pmatrix}1&0\\0&-1\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad M=I$$

$$(c)\quad \begin{pmatrix}0&1\\1&0\end{pmatrix}\quad DET(A_X-\lambda I)=\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}=\lambda^2-1=0\rightsquigarrow\begin{cases}\lambda_1=1\\\lambda_2=-1\end{cases}\rightsquigarrow\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\lambda_1=1\rightsquigarrow\begin{pmatrix}-1&1\\1&-1\end{pmatrix}X=0\quad X_1=\begin{pmatrix}1\\1\end{pmatrix}\quad \lambda_2=-1\rightsquigarrow\begin{pmatrix}1&1\\1&1\end{pmatrix}X=0\quad X_2=\begin{pmatrix}1\\-1\end{pmatrix}\rightsquigarrow M=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$(d)\quad \begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad DET(A_X-\lambda I)=\begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}=\lambda^2+1=0\rightsquigarrow\begin{cases}\lambda_1=i\\\lambda_2=-i\end{cases}\rightsquigarrow\begin{pmatrix}i&0\\0&-i\end{pmatrix}$$

λ DISTINTI ∈ ℂ DIAGONALIZZ. IN ℂ

OSS $X_2=\overline{X_1}$

$$\lambda_1=i\rightsquigarrow\begin{pmatrix}-i&1\\-1&-i\end{pmatrix}X=0\quad X_1=\begin{pmatrix}i\\-1\end{pmatrix}\quad \lambda_2=-i\rightsquigarrow\begin{pmatrix}i&1\\-1&i\end{pmatrix}X=0\quad X_2\begin{pmatrix}i\\1\end{pmatrix}\rightsquigarrow M_{\mathbb{C}}=\begin{pmatrix}i&i\\-1&1\end{pmatrix}$$

FORMA DI JORDAN REALE

$$\begin{pmatrix}a+bi&0\\0&a-bi\end{pmatrix}\rightsquigarrow\begin{pmatrix}a&b\\-b&a\end{pmatrix}\qquad\begin{pmatrix}i&0\\0&-i\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&1\\-1&0\end{pmatrix}\rightsquigarrow M_{\mathbb{R}}=I$$

$$(e)\quad \begin{pmatrix}0&1\\5&0\end{pmatrix}\quad DET(A_X-\lambda I)=\begin{vmatrix}-\lambda&1\\5&-\lambda\end{vmatrix}=\lambda^2-5=0\rightsquigarrow\begin{cases}\lambda_1=2\\\lambda_2=-2\end{cases}\rightsquigarrow\begin{pmatrix}2&0\\0&-2\end{pmatrix}$$

λ DISTINTI ∈ ℝ DIAGONALIZZ. IN ℝ

$$\lambda_1=2\rightsquigarrow\begin{pmatrix}-2&1\\5&-2\end{pmatrix}X=0\quad X_1=\begin{pmatrix}1\\2\end{pmatrix}\quad \lambda_2=-2\rightsquigarrow\begin{pmatrix}2&1\\5&2\end{pmatrix}X=0\quad X_2=\begin{pmatrix}1\\-2\end{pmatrix}\rightsquigarrow M=\begin{pmatrix}1&1\\2&-2\end{pmatrix}$$

$$(f)\quad \begin{pmatrix}0&-1\\5&0\end{pmatrix}\quad DET(A_X-\lambda I)=\begin{vmatrix}-\lambda&-1\\5&-\lambda\end{vmatrix}=\lambda^2+5=0\rightsquigarrow\begin{cases}\lambda_1=2i\\\lambda_2=-2i\end{cases}\rightsquigarrow\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}$$

λ DISTINTI ∈ ℂ DIAGONALIZZ. IN ℂ

OSS $X_2=\overline{X_1}$

$$\lambda_1=2i\rightsquigarrow\begin{pmatrix}-2i&-1\\5&-2i\end{pmatrix}X=0\quad X_1=\begin{pmatrix}2i\\5\end{pmatrix}\quad \lambda_2=-2i\rightsquigarrow\begin{pmatrix}2i&-1\\5&2i\end{pmatrix}X=0\rightsquigarrow X_2=\begin{pmatrix}2i\\-5\end{pmatrix}\rightsquigarrow M_{\mathbb{C}}=\begin{pmatrix}2i&2i\\5&-5\end{pmatrix}$$

FORMA DI JORDAN REALE

$$\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&2\\-2&0\end{pmatrix}\rightsquigarrow M_{\mathbb{R}}=?\qquad M_{\mathbb{R}}^{-1}AM_{\mathbb{R}}=\begin{pmatrix}0&2\\-2&0\end{pmatrix}$$

$$\rightsquigarrow AM_{\mathbb{R}} = M_{\mathbb{R}} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} Ax_1 = -2x_2 \\ Ax_2 = 2x_1 \end{cases} \begin{cases} x_2 = -\frac{1}{2}Ax_1 \\ -\frac{1}{2}A^2x_1 = 2x_1 \end{cases} \begin{cases} x_2 = -\frac{1}{2}Ax_1 \\ A^2x_1 = -4x_1 \end{cases}$$

$$\rightsquigarrow A^2 = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \quad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad M_{\mathbb{R}} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\rightsquigarrow \overset{M_{\mathbb{R}}^{-1}}{-\frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \overset{M_{\mathbb{R}}}{\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}} = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \overset{J_{\mathbb{R}}}{\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

OPPURE (VD. LEZ. 38) $x_1 = \begin{pmatrix} 2i \\ 4 \end{pmatrix} \quad x_2 = \bar{x}_1 = \begin{pmatrix} -2i \\ 4 \end{pmatrix} \rightsquigarrow \tilde{M}_{\mathbb{R}} = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$

$$\overset{M_{\mathbb{R}}^{-1}}{-\frac{1}{8}\begin{pmatrix} 0 & -2 \\ -4 & 0 \end{pmatrix}} \overset{A}{\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}} \overset{M_{\mathbb{R}}}{\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}} = -\frac{1}{8}\begin{pmatrix} -8 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} 0 & -16 \\ 16 & 0 \end{pmatrix} = \overset{J_{\mathbb{R}}}{\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

(g) $\begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix}$ $DET(A-\lambda I) = \begin{vmatrix} 1-\lambda & -3 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda = 0 \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 4 \end{cases} \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ λ DISTINTI $\in \mathbb{R}$ DIAGONALIZZ. IN $\mathbb{R}$

$$\lambda_1 = 0 \rightsquigarrow \begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \lambda_2 = 4 \rightsquigarrow \begin{pmatrix} -3 & -3 \\ -1 & -1 \end{pmatrix} x = 0 \quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

(h) $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ $DET(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0 \quad \lambda_1 = \lambda_2 = 2 \quad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

MG = 1 < MA = 2 JORDANIZZ. 1 BLOCCO $\rightsquigarrow \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad Ax = \lambda x + x_1 \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(i) $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ $DET(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0 \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ λ DISTINTI $\in \mathbb{R}$ DIAGONALIZZ. IN $\mathbb{R}$

$$\lambda_1 = 1 \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda_2 = 3 \rightsquigarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

(l) $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ $DET(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 = 0 \quad \lambda_1 = \lambda_2 = 0 \rightsquigarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

MG = 1 < MA = 2 JORDANIZZ. 1 BLOCCO $\rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad Ax = \lambda x + x_1 \quad x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ O ANCHE $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

(m) $\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ $DET(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ λ DISTINTI $\in \mathbb{R}$ DIAGONALIZZ. IN $\mathbb{R}$

$$\lambda_1 = 1 \rightsquigarrow \begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \lambda_2 = -1 \rightsquigarrow \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

(m) $\begin{pmatrix}1&2\\4&-1\end{pmatrix}$ $DET(A-\lambda I)=\begin{vmatrix}1-\lambda&2\\4&-1-\lambda\end{vmatrix}=\lambda^2-9=0$ $\begin{cases}\lambda_1=3\\ \lambda_2=-3\end{cases}$ $\rightsquigarrow \begin{pmatrix}3&0\\0&-3\end{pmatrix}$ λ DISTINTI $\in \mathbb{R}$ DIAGONALIZZ. IN $\mathbb{R}$

$\lambda_1=3 \rightsquigarrow \begin{pmatrix}-2&2\\4&-4\end{pmatrix}X=0$ $x_1=\begin{pmatrix}1\\1\end{pmatrix}$ $\lambda_2=-3\rightsquigarrow\begin{pmatrix}4&2\\4&2\end{pmatrix}X=0$ $X_2=\begin{pmatrix}1\\-2\end{pmatrix}\rightsquigarrow M=\begin{pmatrix}1&1\\1&-2\end{pmatrix}$

(o) $\begin{pmatrix}1&-2\\5&-1\end{pmatrix}$ $DET(A-\lambda I)=\begin{vmatrix}1-\lambda&-2\\5&-1-\lambda\end{vmatrix}=\lambda^2+9=0$ $\begin{cases}\lambda_1=3i\\ \lambda_2=-3i\end{cases}$ $\rightsquigarrow\begin{pmatrix}3i&0\\0&-3i\end{pmatrix}$ λ DISTINTI $\in \mathbb{C}$ DIAGONALIZZ. IN $\mathbb{C}$

$\lambda_1=3i\rightsquigarrow\begin{pmatrix}1-3i&-2\\5&-1-3i\end{pmatrix}X=0$ $X_1=\begin{pmatrix}2\\1-3i\end{pmatrix}$ $\lambda_2=-3i\rightsquigarrow\begin{pmatrix}1+3i&-2\\5&-1+3i\end{pmatrix}X=0$ $X_2=\begin{pmatrix}1-3i\\5\end{pmatrix}\rightsquigarrow M_{\mathbb{C}}=\begin{pmatrix}2&1-3i\\1-3i&5\end{pmatrix}$

FORMA DI JORDAN REALE $\begin{pmatrix}3i&0\\0&-3i\end{pmatrix}\rightsquigarrow\begin{pmatrix}0&3\\-3&0\end{pmatrix}\rightsquigarrow M_{\mathbb{R}}=?$ $\quad M_{\mathbb{R}}^{-1}AM_{\mathbb{R}}=\begin{pmatrix}0&3\\-3&0\end{pmatrix}$

$\rightsquigarrow AM_{\mathbb{R}}=M_{\mathbb{R}}\begin{pmatrix}0&3\\-3&0\end{pmatrix}\rightsquigarrow\begin{cases}AX_1=-3X_2\\AX_2=3X_1\end{cases}$ $\begin{cases}X_2=-\frac{1}{3}AX_1\\-\frac{1}{3}A^2X_1=3X_1\end{cases}$ $\begin{cases}X_2=-\frac{1}{3}AX_1\\A^2X_1=-9X_1\end{cases}$

$\rightsquigarrow A^2=\begin{pmatrix}1&-2\\5&-1\end{pmatrix}\begin{pmatrix}1&-2\\5&-1\end{pmatrix}=\begin{pmatrix}-9&0\\0&-9\end{pmatrix}$ $X_1=\begin{pmatrix}3\\0\end{pmatrix}$ $X_2=\begin{pmatrix}-1\\-5\end{pmatrix}$ $M_{\mathbb{R}}=\begin{pmatrix}3&-1\\0&-5\end{pmatrix}$

$\rightsquigarrow \underbrace{\frac{-1}{15}\begin{pmatrix}-5&1\\0&3\end{pmatrix}}_{M_{\mathbb{R}}^{-1}}\begin{pmatrix}1&-2\\5&-1\end{pmatrix}\underbrace{\begin{pmatrix}3&-1\\0&-5\end{pmatrix}}_{M_{\mathbb{R}}}=\frac{-1}{15}\begin{pmatrix}0&9\\15&-3\end{pmatrix}\begin{pmatrix}3&-1\\0&-5\end{pmatrix}=\frac{-1}{15}\begin{pmatrix}0&-45\\45&0\end{pmatrix}=\begin{pmatrix}0&3\\-3&0\end{pmatrix}$

OPPURE (LEZ.58) $X_1=\begin{pmatrix}2\\1-3i\end{pmatrix}$ $X_2=\overline{X_1}=\begin{pmatrix}2\\1+3i\end{pmatrix}\rightsquigarrow(2,1)\pm i(0,3)\rightsquigarrow M_{\mathbb{R}}=\begin{pmatrix}2&0\\1&3\end{pmatrix}$

(p) $\begin{pmatrix}3&0\\3&3\end{pmatrix}$ $DET(A-\lambda I)=\begin{vmatrix}3-\lambda&0\\3&3-\lambda\end{vmatrix}=(3-\lambda)^2=0$ $\lambda_1=\lambda_2=3\rightsquigarrow\begin{pmatrix}0&0\\3&0\end{pmatrix}X=0$ $X_1=\begin{pmatrix}0\\3\end{pmatrix}$

MG=1 < MA=2 JORDANIZZ. 1 BLOCCO $\rightsquigarrow\begin{pmatrix}3&1\\0&3\end{pmatrix}$ $\quad AX=\lambda X+X_1$ $X_2=\begin{pmatrix}1\\0\end{pmatrix}$ $\rightsquigarrow M=\begin{pmatrix}0&1\\3&0\end{pmatrix}$

(q) $\begin{pmatrix}3&3\\3&3\end{pmatrix}$ $DET(A-\lambda I)=\begin{vmatrix}3-\lambda&3\\3&3-\lambda\end{vmatrix}=\lambda^2-6\lambda=0$ $\begin{cases}\lambda_1=0\\ \lambda_2=6\end{cases}$ $\rightsquigarrow\begin{pmatrix}0&0\\0&6\end{pmatrix}$ λ DISTINTI $\in\mathbb{R}$ DIAGONALIZZ. IN $\mathbb{R}$

$\lambda_1=0\rightsquigarrow X_1=\begin{pmatrix}1\\-1\end{pmatrix}$ $\lambda_2=6\rightsquigarrow\begin{pmatrix}-3&3\\3&-3\end{pmatrix}X=0$ $X_2=\begin{pmatrix}1\\1\end{pmatrix}\rightsquigarrow M=\begin{pmatrix}1&1\\-1&1\end{pmatrix}$

(r) $\begin{pmatrix}-1&2\\8&-1\end{pmatrix}$ $DET(A-\lambda I)=\begin{vmatrix}-1-\lambda&2\\8&-1-\lambda\end{vmatrix}=\lambda^2+2\lambda-15=0$ $\begin{cases}\lambda_1=-5\\ \lambda_2=3\end{cases}$ $\rightsquigarrow\begin{pmatrix}-5&0\\0&3\end{pmatrix}$ λ DISTINTI $\in\mathbb{R}$ DIAGONALIZZ. IN $\mathbb{R}$

$\lambda_1=-5\rightsquigarrow\begin{pmatrix}4&2\\8&4\end{pmatrix}X=0$ $X_1=\begin{pmatrix}1\\-2\end{pmatrix}$ $\lambda_2=3$ $\begin{pmatrix}-4&2\\8&-4\end{pmatrix}X=0$ $X_2=\begin{pmatrix}1\\2\end{pmatrix}\rightsquigarrow M=\begin{pmatrix}1&1\\-2&2\end{pmatrix}$

(ς) $\begin{pmatrix}-1 & 2\\ -8 & -1\end{pmatrix}$ $\begin{vmatrix}-1-\lambda & 2\\ -8 & -1-\lambda\end{vmatrix} = \lambda^2+2\lambda+17=0$ $\lambda = \frac{-2\pm\sqrt{4-68}}{2}$ $\begin{cases}\lambda_1=-1+4i\\ \lambda_2=-1-4i\end{cases} \rightsquigarrow \begin{pmatrix}-1+4i & 0\\ 0 & -1-4i\end{pmatrix}$ λ DISTINTI ∈ ℂ DIAGONALIZZ. IN ℂ

AUTOVETT. CONIUGATI

$\lambda_1=-1+4i \rightsquigarrow \begin{pmatrix}-4i & 2\\ -8 & -4i\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}2\\ 4i\end{pmatrix} \quad x_2=\bar{x}_1=\begin{pmatrix}2\\ -4i\end{pmatrix} \rightsquigarrow M_{\mathbb{C}}=\begin{pmatrix}2 & 2\\ 4i & -4i\end{pmatrix}$

FORMA DI JORDAN REALE $\begin{pmatrix}a+bi & 0\\ 0 & a-bi\end{pmatrix} \rightsquigarrow \begin{pmatrix}a & b\\ -b & a\end{pmatrix}, \begin{pmatrix}-1+4i & 0\\ 0 & -1-4i\end{pmatrix} \rightsquigarrow \begin{pmatrix}-1 & 4\\ -4 & -1\end{pmatrix} \rightsquigarrow$ (L.58) $\hat{M}_{\mathbb{R}}=\begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix}$

OPPURE

$\rightsquigarrow AM_{\mathbb{R}}=M_{\mathbb{R}}\begin{pmatrix}-1 & 4\\ -4 & -1\end{pmatrix} \rightsquigarrow \begin{cases}Ax_1=-x_1-4x_2\\ Ax_2=4x_1-x_2\end{cases} \quad \begin{cases}(A+I)x_1=-4x_2\\ (A+I)x_2=4x_1\end{cases} \quad \begin{cases}4x_2=-(A+I)x_1\\ (A+I)4x_2=16x_1\end{cases}$

$\rightsquigarrow -(A+I)^2x_1=16x_1 \quad (A+I)^2=\begin{pmatrix}0 & 2\\ -8 & 0\end{pmatrix}^2=\begin{pmatrix}-16 & 0\\ 0 & -16\end{pmatrix}=-16I \rightsquigarrow 16Ix_1=16x_1$???

FISSIAMO x_1 ARBITRARIO $\quad x_1=\begin{pmatrix}1\\ 0\end{pmatrix} \quad (A+I)x_1=\begin{pmatrix}0\\ -8\end{pmatrix}=-4x_2 \rightsquigarrow x_2=\begin{pmatrix}0\\ 2\end{pmatrix} \rightsquigarrow M_{\mathbb{R}}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$

VER. $\underbrace{\frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}}_{M_{\mathbb{R}}^{-1}}\begin{pmatrix}-1 & 2\\ -8 & -1\end{pmatrix}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}}_{M_{\mathbb{R}}}=\frac{1}{2}\begin{pmatrix}-2 & 4\\ -8 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-2 & 8\\ -8 & -2\end{pmatrix}=\begin{pmatrix}-1 & 4\\ -4 & -1\end{pmatrix}$ OK

(δ) $\begin{pmatrix}0 & 3\\ 1 & 0\end{pmatrix}$ $DET(A-\lambda I)=\begin{vmatrix}-\lambda & 3\\ 1 & -\lambda\end{vmatrix}=\lambda^2-3=0$ $\begin{cases}\lambda_1=\sqrt{3}\\ \lambda_2=-\sqrt{3}\end{cases} \rightsquigarrow \begin{pmatrix}\sqrt{3} & 0\\ 0 & -\sqrt{3}\end{pmatrix}$ λ DISTINTI ∈ ℝ DIAGONALIZZ. IN ℝ

$\lambda_1=\sqrt{3} \quad \begin{pmatrix}-\sqrt{3} & 3\\ 1 & -\sqrt{3}\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix} \quad \lambda_2=-\sqrt{3} \quad \begin{pmatrix}\sqrt{3} & 3\\ 1 & \sqrt{3}\end{pmatrix}X=0 \quad x_2=\begin{pmatrix}\sqrt{3}\\ -1\end{pmatrix} \rightsquigarrow M=\begin{pmatrix}\sqrt{3} & \sqrt{3}\\ 1 & -1\end{pmatrix}$

(μ) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $\begin{vmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{vmatrix}=\lambda^2-5\lambda-2=0$ $\lambda=\frac{5\pm\sqrt{33}}{2}$ $\begin{cases}\lambda_1=1/2(5+\sqrt{33})\\ \lambda_2=1/2(5-\sqrt{33})\end{cases} \rightsquigarrow \begin{pmatrix}\frac{1}{2}(5+\sqrt{33}) & 0\\ 0 & \frac{1}{2}(5-\sqrt{33})\end{pmatrix}$

$\begin{cases}\lambda_1=1/2(5+\sqrt{33}) \rightsquigarrow \begin{pmatrix}-\frac{3}{2}-\frac{\sqrt{33}}{2} & 2\\ 3 & \frac{3}{2}-\frac{\sqrt{33}}{2}\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}3/2-\sqrt{33}/2\\ -3\end{pmatrix}\\ \lambda_2=1/2(5-\sqrt{33}) \rightsquigarrow \begin{pmatrix}-\frac{3}{2}+\frac{\sqrt{33}}{2} & 2\\ 3 & \frac{3}{2}+\frac{\sqrt{33}}{2}\end{pmatrix}X=0 \quad x_2=\begin{pmatrix}2\\ 3/2-\sqrt{33}/2\end{pmatrix}\end{cases} \rightsquigarrow M=\begin{pmatrix}\frac{3-\sqrt{33}}{2} & 2\\ -3 & \frac{3-\sqrt{33}}{2}\end{pmatrix}$

(ν) $\begin{pmatrix}1 & 2\\ -3 & 4\end{pmatrix}$ $\begin{vmatrix}1-\lambda & 2\\ -3 & 4-\lambda\end{vmatrix}=\lambda^2-5\lambda+10=0$ $\lambda=\frac{5\pm\sqrt{-15}}{2}$ $\begin{cases}\lambda_1=\frac{5+i\sqrt{15}}{2}\\ \lambda_2=\frac{5-i\sqrt{15}}{2}\end{cases} \rightsquigarrow \begin{pmatrix}(5+i\sqrt{15})/2 & 0\\ 0 & (5-i\sqrt{15})/2\end{pmatrix}$

$\lambda_1=1/2(5+i\sqrt{15}) \rightsquigarrow \begin{pmatrix}-3/2-i\sqrt{15}/2 & 2\\ -3 & \frac{3}{2}-i\frac{\sqrt{15}}{2}\end{pmatrix}X=0 \quad x_1=\begin{pmatrix}\frac{3}{2}-i\frac{\sqrt{15}}{2}\\ -3\end{pmatrix} \rightsquigarrow x_2=\bar{x}_1=\begin{pmatrix}\frac{3}{2}+i\frac{\sqrt{15}}{2}\\ -3\end{pmatrix}$

$\rightsquigarrow M=\begin{pmatrix}3/2-i\sqrt{15}/2 & 3/2+i\sqrt{15}/2\\ -3 & -3\end{pmatrix}$

2. Determinare per quali valori del parametro reale a la matrice $\begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$

(a) è diagonalizzabile sui reali,
(b) è diagonalizzabile sui reali mediante una matrice ortogonale,
(c) ha il ker non banale,
(d) ammette l'autovalore $\lambda = 1$,
(e) ammette l'autovalore $\lambda = 2 + i$,
(f) ammette $(1, 5)$ come autovettore,
(g) non è diagonalizzabile sui complessi.

(a) $A = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$ $\quad DET(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ a & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 - 2a = 0$

$\lambda = \dfrac{4 \pm \sqrt{16 - 12 + 8a}}{2} = 2 \pm \sqrt{1+2a}$ $\quad \lambda \in \mathbb{R} \Rightarrow 1 + 2a \geq 0 \quad a \geq -\frac{1}{2}$

$a = -\frac{1}{2}$ $\quad \begin{pmatrix} 1 & 2 \\ -1/2 & 3 \end{pmatrix}$ $\quad (A - 2I)X = \begin{pmatrix} -1 & 2 \\ -1/2 & 1 \end{pmatrix}$ $\quad MG = 1 < MA = 2$

NON DIAGONALIZZ.

$\Rightarrow$ A DIAGONALIZZ. IN $\mathbb{R}$ PER $a > -1/2$

(b) TEOREMA SPETTRALE $\Rightarrow A = A^T \Rightarrow a = 2$

(c) $AX = 0 \quad X \neq 0 \iff \det A = 3 - 2a = 0 \rightsquigarrow a = 3/2$

(d) $\lambda_1 = 2 + \sqrt{1+2a} \geq 2 \qquad \lambda_2 = 2 - \sqrt{1+2a} \Rightarrow 1 + 2a = 1 \qquad a = 0$

(e) $\lambda_1 = 2 + \sqrt{1+2a} = 2 + i \Rightarrow 1 + 2a = -1 \qquad a = -1$

(f) $\begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 5 \end{pmatrix} \rightsquigarrow \begin{cases} 1 + 10 = \lambda \\ a + 15 = 5\lambda \end{cases} \quad \begin{cases} \lambda = 11 \\ a = 55 - 15 \end{cases} \rightsquigarrow a = 40$

VER. $\begin{cases} \lambda_1 = 11 \\ \lambda_2 = -7 \end{cases} \quad \begin{pmatrix} -10 & 2 \\ 40 & -8 \end{pmatrix} X = 0 \rightsquigarrow X = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

(g) $a > -1/2 \rightsquigarrow$ A DIAGONALIZZABILE IN $\mathbb{R} \subseteq \mathbb{C}$ ($\lambda \in \mathbb{R}$ DISTINTI)

$a = -1/2 \rightsquigarrow$ A NON DIAGONALIZZABILE NÉ IN $\mathbb{R}$ NÉ IN $\mathbb{C}$ $\quad J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

JORDANIZZABILE

$a < -1/2 \rightsquigarrow$ A DIAGONALIZZABILE IN $\mathbb{C}$ MA NON IN $\mathbb{R}$ ($\lambda \in \mathbb{C}$ DISTINTI)

3. Determinare per quali valori del parametri reali a e b la matrice $\begin{pmatrix} a & 1 \\ b & 2 \end{pmatrix}$

(a) ammette autovalori immaginari puri,

(b) ammette l'autovalore $\lambda = 3 + 5i$,

(c) è diagonalizzabile sui reali mediante una matrice ortogonale e rappresenta un'applicazione lineare non surgettiva,

(d) ammette $\lambda = 7$ come autovalore con autovettore corrispondente $(1, 1)$,

(e) ammette $\lambda = -1$ come autovalore ma non è diagonalizzabile sui complessi,

(f) è simile alla matrice $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) $A = \begin{pmatrix} a & 1 \\ b & 2 \end{pmatrix}$ $\quad DET(A-\lambda I) = \begin{vmatrix} a-\lambda & 1 \\ b & 2-\lambda \end{vmatrix} = (a-\lambda)(2-\lambda) - b =$

$\lambda^2 - (a+2)\lambda + 2a - b = 0 \qquad \lambda = \dfrac{(a+2) \pm \sqrt{(a+2)^2 - 4(2a-b)}}{2}$

$\rightsquigarrow a+2=0 \quad a=-2 \quad \lambda = \pm \frac{1}{2}\sqrt{4b+16} = \pm\sqrt{b+4} \rightsquigarrow b<-4$

(b) $\lambda = 3+5i \rightsquigarrow \dfrac{a+2}{2} = 3 \quad a=4 \rightsquigarrow \lambda = 3 + \frac{1}{2}\sqrt{36-32+4b} = 3+\sqrt{1+b}$

$\rightsquigarrow 1+b = -25 \quad b=-26$

(c) TEOREMA SPETTRALE $\Rightarrow A = A^T \Rightarrow b=1$

NON SURGETTIVA $\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \delta \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \delta = 1/2 \quad a = 1/2$

(d) $\begin{pmatrix} a & 1 \\ b & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{cases} a+1=7 \\ b+2=7 \end{cases} \rightsquigarrow \begin{cases} a=6 \\ b=5 \end{cases}$

(e) $\lambda^2 - (a+2)\lambda + 2a - b = 0, \ \lambda=-1 \rightsquigarrow 1 + a + 2 + 2a - b = 0 \quad 3a - b + 3 = 0$

$b = 3a+3 \rightsquigarrow \Delta = (a+2)^2 - 4(2a-b) = a^2+4a+4-4(2a-3a-3) =$

$= a^2+4a+4+4a+12 = a^2+8a+16 = (a+4)^2 = 0$

$\Rightarrow a=-4 \qquad b=-9$

VER. $\begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix}$ $\quad DET(A-\lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ -9 & 2-\lambda \end{vmatrix} = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$

$\lambda_1 = \lambda_2 = -1 \rightsquigarrow \begin{pmatrix} 3 & 1 \\ -9 & 3 \end{pmatrix} x = 0 \quad x_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \rightsquigarrow J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

JORDANIZZABILE

(f) $M^{-1}AM = B = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \Rightarrow tr(A) = tr(B) \quad det(A) = det(B)$

$$\rightsquigarrow \begin{cases} a+2=5 \\ 2a-b=-2 \end{cases} \quad \begin{cases} a=3 \\ b=8 \end{cases} \rightsquigarrow \begin{pmatrix}3 & 1\\ 8 & 2\end{pmatrix}$$

DETERMINAZIONE DELLA MATRICE DI CAMBIO BASE $A \to B$

$$M^{-1}AM = B = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \Leftrightarrow AM = MB$$

$$M = \begin{pmatrix}| & |\\ x_1 & x_2\\ | & |\end{pmatrix} \rightsquigarrow \begin{cases} Ax_1 = x_1 + 3x_2 \\ Ax_2 = 2x_1 + 4x_2 \end{cases} \rightsquigarrow \begin{cases} (A-I)x_1 = 3x_2 \\ (A-4I)x_2 = 2x_1 \end{cases}$$

$$\rightsquigarrow (A-4I)\,3x_2 = 6x_1 \rightsquigarrow (A-4I)(A-I)\,x_1 = 6x_1$$

$$(A-4I)(A-I) = A^2 - A - 4A + 4I = A^2 - 5A + 4I$$

$$A^2 = \begin{pmatrix}17 & 5\\ 40 & 12\end{pmatrix} \quad 5A = \begin{pmatrix}15 & 5\\ 40 & 10\end{pmatrix} \quad 4I \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix}6 & 0\\ 0 & 6\end{pmatrix} x_1 = 6x_1 \rightsquigarrow \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} x_1 = 0 \qquad x_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$(A-I) = \begin{pmatrix}2 & 1\\ 8 & 1\end{pmatrix} \quad (A-I)x_1 = \begin{pmatrix}2\\ 8\end{pmatrix} = 3x_2 \rightsquigarrow x_2 = \frac{1}{3}\begin{pmatrix}2\\ 8\end{pmatrix}$$

$$\rightsquigarrow M = \begin{pmatrix}3 & 2\\ 0 & 8\end{pmatrix} \qquad M^{-1} = \frac{1}{24}\begin{pmatrix}8 & -2\\ 0 & 3\end{pmatrix}$$

$$\overset{M^{-1}}{\frac{1}{24}\begin{pmatrix}8 & -2\\ 0 & 3\end{pmatrix}} \overset{A}{\begin{pmatrix}3 & 1\\ 8 & 2\end{pmatrix}} \overset{M}{\begin{pmatrix}3 & 2\\ 0 & 8\end{pmatrix}} = \frac{1}{24}\begin{pmatrix}8 & 4\\ 24 & 6\end{pmatrix} \overset{A}{\begin{pmatrix}3 & 2\\ 0 & 8\end{pmatrix}} = \frac{1}{24}\begin{pmatrix}24 & 48\\ 72 & 96\end{pmatrix} = \overset{B}{\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}}$$

METODO ALTERNATIVO

$$M_A, M_B \text{ t.c. } M_A^{-1} A M_A = M_B^{-1} B M_B = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

$\lambda_1, \lambda_2 \to$ Es. "1a"

$$\rightsquigarrow M_B^{-1} B M_B = M_A^{-1} A M_A \rightsquigarrow B = M_B M_A^{-1} A M_A M_B^{-1}$$

$$M_A M_B^{-1}$$