1. Let us consider the functional

$$F(u) = \int_0^1 (\dot{u}^2 + \dot{u} + x^3 u)\ dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) + u(1) = 3$.

(b) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) - u(1) = 3$.

See CdV 2019_5 Ex 1

— o — o —

2. For every $f \in L^2((0,1))$, let us consider the Dirichlet problem

$$u'' = u^3 + \sin u + f(x), \qquad u(0) = u(1) = 0.$$

(a) Prove that the problem admits a unique solution.

(b) Discuss the regularity of this solution.

(c) Let $S : L^2((0,1)) \to L^2((0,1))$ be the operator that associates to each function f the corresponding solution u. Determine whether S is a compact operator.

(a) Let us consider the Lagrangian $L(x,s,p) = \frac{1}{2}p^2 + \frac{1}{4}s^4 - \cos s + f(x)s$ and the problem

$$\min\Big\{ \underbrace{\int_0^1 L(x,u,\dot u)\,dx}_{F(u)} : u \in H^1((0,1)),\ u(0) = u(1) = 0 \Big\}.$$

Classical direct method $\Rightarrow$ $\boxed{\text{existence}}$. The key estimate for compactness is

$$F(u) \geq \int_0^1 \frac{1}{2}\dot u^2 - 1 - |f(x)|\cdot|u| \geq \frac{1}{2}\|\dot u\|_2^2 - 1 - \|f\|_2 \cdot \|u\|_2$$

$$\geq \frac{1}{2}\|\dot u\|^2 - 1 - c\|f\|_2 \cdot \|\dot u\|_2$$

Poincaré or L^∞ estimate

$\boxed{\text{Uniqueness}}$ follows from the strict convexity of $L(x,s,p)$ wrt (s,p) (because the function $s^3 + \sin s$ is strictly increasing).

(b) From minimality we obtain in the usual way that $(\dot u)' = u^3 + \sin u + f(x)$, where the LHS is the weak derivative of $\dot u$. Since $u \in C^0$, the RHS $\in L^2$ and therefore $\dot u \in H^1$ which means $\boxed{u \in H^2}$.

(c) $\boxed{\text{YES}}$ The operator S is compact. Indeed, let $\{f_n\} \subseteq L^2$ be a bounded sequence, and let $\{u_n\}$ be the corresponding solutions to the minimum problem. We observe that the min. values are ≤ 0, because $u \equiv 0$ is always a competitor. Then as in the key estimate we find that

$$0 \geq F(u_n) \geq \frac{1}{2}\|\dot u_n\|_2^2 - 1 - c\|f_n\|_2 \|\dot u_n\|_2$$

$$\geq \frac{1}{2}\|\dot u_n\|_2^2 - 1 - M\|\dot u_n\|_2.$$

This provides a uniform bound on $\|\dot u_n\|$. Due to the DBC, this shows that $\{u_n\}$ is relatively compact in C^0, hence also in L^2.

— o — o —

3. Let d be a positive integer, and let B_d denote the unit ball in $\mathbb{R}^d$ with center in the origin. For every real number $m > 0$, let us set

$$I_d(m) := \inf\left\{\underbrace{\int_{B_d} (u^{19} + \arctan(u^2))\,dx}_{F(u)} : u \in C_c^1(B_d),\ \int_{B_d} \|\nabla u(x)\|^7\,dx \le m\right\}.$$

(a) In dimension $d = 3$, determine whether there exists $m > 0$ such that $I_3(m) = 0$.

(b) Determine for which values of d it turns out that $I_d(m)$ is a real number for every $m > 0$.

(a) $\boxed{\text{YES}}$ Let us observe that there exist $r > 0$ such that

$$s^{19} + \arctan(s^2) \ge 0 \qquad \forall\ |s| \le r$$

Now in dimension $d = 3$ it turns out that $W^{1,7} \hookrightarrow L^\infty$ and

$$\|u\|_{L^\infty} \underset{\text{Sobolev embedd.}}{\le} C_1 (\|u\|_{L^7} + \|\nabla u\|_{L^7}) \underset{\text{Poincaré}}{\le} C_2 \|\nabla u\|_{L^7}$$

Therefore, if m is small enough it turns out that $|u(x)| \le r$ for every $x \in B$, and therefore the integral is always ≥ 0 (and $= 0$ when $u = 0$, of course).

(b) $I_d(m)$ is a real number for every $m > 0$ iff $\boxed{d \le 11}$, namely iff $19 < 7^*$. This follows from two facts.

- If $19 < 7^*$, then $\|u\|_{L^{19}} \le C_1 \|u\|_{W^{1,7}} \le C_2 \|\nabla u\|_{L^7}$ as before, and therefore

$$F(u) \ge -\|u\|_{L^{19}}^{19} \ge -C_3 \|\nabla u\|_{L^7}^{19} \ge -C_3\, m^{19/7}$$

- If $19 > 7^*$, then there exists a nonnegative function u that belongs to $W_0^{1,7} \setminus L^{19}$. If $\varepsilon > 0$ is small enough, then $F(-\varepsilon u) = -\infty$ and $\|\nabla(-\varepsilon u)\|_{L^7}^7 \le m$. A suitable approx of $-\varepsilon u$ shows that the inf is $-\infty$

Alternative for the last point: take $u \in C_c^1(B_d)$ with $u \le 0$ but $u \not\equiv 0$. Then there exists an exponent $a > 0$ such that, setting $u_n(x) = n^a u(nx)$, it turns out that

$$F(u_n) \to -\infty \qquad \text{and} \qquad \int \|\nabla u_n\|^* \to 0.$$

— o — o —

4. For every $f : (1, +\infty) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^4) \qquad \forall x \in (1, +\infty).$$

Determine for which real numbers $p \geq 1$ the restriction of T defines

(a) a *continuous* operator $L^p((1, +\infty) \to L^2((1, +\infty))$,
(b) a *continuous* operator $L^2((1, +\infty)) \to L^p((1, +\infty))$,
(c) a *compact* operator $H^1((1, +\infty)) \to L^p((1, +\infty))$.

converges $\iff \frac{3p}{4(p-2)} > 1$ $\iff p < 8$

(a) $\boxed{2 \le p < 8}$ Indeed

- if $2 \le p < 8$ then

$$\int_1^{+\infty} f(x^4)^2\,dx \underset{x^4 = y}{=} \int_1^{+\infty} \underbrace{f(y)^2}_{\frac{2}{p}} \underbrace{\frac{1}{4y^{3/4}}}_{\frac{p-2}{p}}\,dy \underset{\text{Hölder}}{\le} \frac{1}{4}\left\{\int_1^{+\infty} |f(y)|^p\,dy\right\}^{\frac{2}{p}} \left\{\int_1^{+\infty} \frac{1}{y^{\frac{3}{4}\frac{p}{p-2}}}\right\}^{\frac{p-2}{p}}$$

- if $p < 2$ then $f(x) = \dfrac{1}{x^2\sqrt{x-1}}$ is a counterexample

- If $p \ge 8$ then $f(x) = \dfrac{1}{\sqrt[8]{x}\,\sqrt{\log(x+1)}}$ is a counterexample.

(b) $\boxed{p \le 2}$ Indeed

- If $p < 2$ then as before

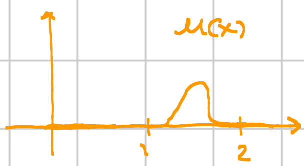
$$\int_1^{+\infty} |f(x^4)|^p\,dx = \int_1^{+\infty} \underbrace{|f(y)|^p}_{\frac{p}{2}} \frac{1}{4} \underbrace{\frac{1}{y^{3/4}}}_{\frac{2-p}{2}}\,dy \le \frac{1}{4}\left\{\int_1^{+\infty} f(y)^2\right\}^{\frac{p}{2}} \Bigg\{\underbrace{\int_1^{+\infty} \frac{1}{y^{\frac{3}{4}\frac{2}{2-p}}}\,dy}_{\text{convergent if } 1 \le p \le 2}\Bigg\}^{\frac{2-p}{2}}$$

- If $p = 2$, the previous inequality is easier: $|f(y)|^2 \dfrac{1}{y^{3/4}} \le |f(y)|^2$.

- If $p > 2$ then $f(x) = \dfrac{1}{(x-1)^{1/2}\,|\log(x-1)|}$ is a counterexample

(c) The operator is $\boxed{\text{NEVER}}$ compact.
Consider $u : (1, +\infty) \to [0, +\infty)$ be a nonzero function of class C^1 with support $\subseteq (1, 2)$, and set

$$u_n(x) = u(x - n)$$



Then $\{u_n\}$ is bounded in $H^1((1, +\infty))$, but any subsequence cannot be a Cauchy sequence in $L^p((1, +\infty))$ because the supports of u_n and u_m are disjoint if $n \neq m$.

This is the solution that was originally posted. Everything after "consider" is correct, but it proves only that the embedding $H^1((1, +\infty)) \to L^p((1, +\infty))$ is NOT compact.

(c) This point resulted to be harder than expected. In any case, the solution is very instructive.

The operator is compact for every $\boxed{p \geq 1}$ (but not for $p = +\infty$, as one can verify by setting $u_m(x) = \varphi(x-m)$, where $\varphi \in C_c^\infty((0,1))$ is any function that does not vanish identically).

Let us take any bounded sequence $\{f_n\} \subseteq H^1((1,+\infty))$, and let us show that $\{f_n(x^4)\}$ is relatively compact in $L^p((1,+\infty))$.

① The sequence $\{f_n(x)\}$ is equibounded in $L^\infty((1,+\infty))$

(bound in L^2 + uniform $\frac{1}{2}$ - Hölder continuity)

As a consequence, $\{f_n(x^4)\}$ is equibounded in $L^\infty((1,+\infty))$

② The sequence $\{f_n(x^4)\}$ is equibounded in $L^1((1,+\infty))$ (point (b))

③ The sequence $\{f_n(x^4)\}$ is equibounded in $L^p((1,+\infty))$ (① + ②)

④ For every $T>0$ the sequence $\{f_n(x^4)\}$ is equibounded in $H^1((0,T))$, and hence relatively compact in $C^0([0,T])$ (Ascoli-Arzelà)

⑤ There exists $f_{n_k}(x^4) \to g_\infty(x)$ unif. on compact subsets of $[1,+\infty)$ (point ④ + diagonal argument)

⑥ For every $T>1$ it turns out that

$$\int_T^{+\infty} |f_n(x^4)|^p \, dx = \frac{1}{4}\int_{T^4}^{+\infty} |f_n(y)|^p \frac{1}{y^{3/4}} \, dy \leq \frac{1}{4\,T^{3/4}} \int_1^{+\infty} |f_n(y)|^p \, dy \leq \frac{M}{T^{3/4}}$$

⑦ Also $\displaystyle\int_T^{+\infty} |g_\infty(x)|^p \, dx \leq \frac{M}{T^{3/4}}$ (Point ⑥ + Fatou Lemma)

⑧ We show that $f_n(x^4) \to g_\infty(x)$ in $L^p((1,+\infty))$

$$\int_1^{+\infty} |f_n(x^4) - g_\infty(x)|^p \, dx \leq \underbrace{\int_1^{T} |f_n(x^4) - g_\infty(x)|^p \, dx}_{\downarrow \text{ point ④ } \; 0} + \underbrace{\int_T^{+\infty} |\ldots|^p}_{\leq \frac{M'}{T^{3/4}} \text{ point ⑥ + ⑦}}$$