1. Prove that there exists a sequence of functions $u_n : [0,1] \to \mathbb{R}$ of class C^∞ such that

- $\{u_n\}$ is an orthonormal basis of $L^2((0,1))$,
- $u_n(0) = u_n(1) = 0$ for every positive integer n,
- for every positive integer n, there exists a negative real number λ_n such that

$$(\cos x \cdot u_n'(x))' = \lambda_n u_n(x) \qquad \forall x \in [0,1].$$

The idea is the following: for every $f \in L^2((0,1))$, there exists a unique $u \in H^2((0,1))$ such that $(\cos x \cdot u')' = f$ and $u(0) = u(1) = 0$.
Moreover, the operator $Tf = u$ is symmetric, linear, and compact from L^2 to L^2. At that point the conclusion follows from the spectral theorem.

- Existence and uniqueness of u can be proved in at least 3 different ways:
  - $\to$ Lax-Milgram
  - $\to$ direct method via variational formulation

$$\min\left\{ \int_0^1 \left(\frac{1}{2}\cos x \cdot \dot{u}^2 + uf\right) : u \in H^1((0,1)),\ u(0) = u(1) = 0 \right\}$$

  - $\to$ explicit formula

$$u'(x) = \frac{1}{\cos x}\left(\int_0^x f(t)\,dt + c\right) \rightsquigarrow u(x) = \int_0^x \frac{1}{\cos s}\left(\int_0^s f(t)\,dt + \underset{\substack{\text{to be chosen so} \\ \text{that } u(1)=0}}{c}\right) ds$$

The crucial point is always that $\cos x$ is bounded from below by a positive constant in $[0,1]$.

- Symmetry. Let $u = Tf$ and $v = Tg$. Then

$$\langle Tf, g\rangle = \int_0^1 u(x)\,g(x) = \int_0^1 u(x)\,[\cos x\, v'(x)]' = -\int_0^1 u'(x)\cos x \cdot v'(x)$$

integr. by parts + DBCs on u

$$= \int_0^1 [\cos x\, u'(x)]'\,v(x) = \int_0^1 f(x)\,v(x) = \langle f, Tg\rangle$$

integr. by parts + DBCs on v.

- Compactness. If $\{f_m\}$ is bounded in L^2, then $\{\cos x \cdot \dot{u}_m\}$ has a subsequence that converges uniformly (Ascoli-Arzelà thm: they are equi $\frac{1}{2}$-Hölder and equibounded because $\dot{u}_m$ vanishes somewhere due to Rolle's thm.). It follows that $\{\dot{u}_m\}$ converges unif. (up to subsequences), and the same for $\{u_m\}$.
- Sign of eigenvalues. If $(\cos x \cdot u'(x))' = \lambda u(x)$, then

$$\lambda \int_0^1 u(x)^2 = \int_0^1 (\cos x \cdot u'(x))'\,u(x) = -\int_0^1 \cos x \cdot u'(x)^2 < 0.$$

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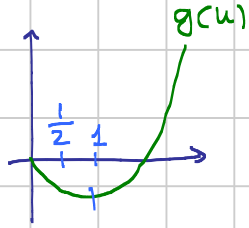
2. Discuss existence, uniqueness and regularity of solutions to the boundary value problem

$$u'' = -1 + \sqrt{u}, \qquad u(0) = 1/2, \quad \dot{u}(2020) = 1.$$

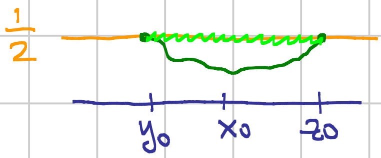
Let us set $\displaystyle F(u) := \int_0^{2020} \Big(\frac{1}{2}\dot{u}^2 - \dot{u} \underbrace{- u + \frac{2}{3} u^{3/2}}_{g(u)} \Big)$, and let us consider the min. pblm.

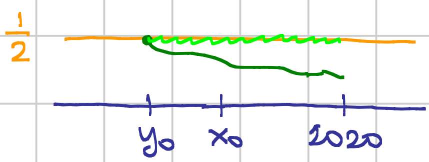
$\min \{ F(u) : u \in H^1((0,2020)),\ u(0) = \frac{1}{2},\ u(x) \geq 0 \ \forall x \in [0,2020] \}$



- <u>Existence</u>: standard direct method.
  Compactness follows from the bounds
  $$L(x,s,p) \geq \frac{1}{4}p^2 - A \qquad + \text{DBC in } x=0$$
  LSC from LSC of the norm and the continuity of $g(s)$.
  Both the DBC and the constraint $u(x) \geq 0$ pass to the limit

- <u>Truncation argument</u>: any min. point u_0 actually satisfies $u(x) \geq \frac{1}{2}$ in $[0,2020]$. Indeed, let us assume by contradiction that $u(x_0) < \frac{1}{2}$.
  Then there are two cases





  In both cases the truncation is better. This is trivial for the terms $\int \dot{u}_0^2$ and $\int g(u)$, and then we observe that $\int_{y_0}^{z_0} \dot{u}_0 = 0$ in the first case and $\int_{y_0}^{2020} \dot{u}_0 < 0$ in the second case.

- ELE and regularity: $\boxed{\text{AFTER}}$ the truncation argument we are allowed to compute ELE in the usual way. We deduce that any min. point is a solution to the diff. equ. and satisfies the NBC in $x = 2020$. Then a bootstrap argument leads to C^∞.

- Uniqueness. Usual argument
  $\to$ any sol. to the equ. is a min. point of the functional
  $\to$ min. points are unique
  Both facts follow from the strict convexity of the Lagrangian.

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3. For every positive real number R, let B_R denote the ball in $\mathbb{R}^3$ with center in the origin and radius R. For every real number $p > 1$, and every real number $r \in (0,1)$, let us set

$$I(p,r) := \inf \left\{ \int_{B_1 \setminus B_r} \left(|\nabla u|^p + u^{2020} \right) dx : u \in C^\infty(B_1),\ u(x) = 1 \text{ for every } x \in B_r \right\}.$$

(a) Prove that $I(p,r) > 0$ for every $p > 1$ and every $r \in (0,1)$,

(b) Prove that for every $p > 1$ there exists

$$\ell(p) := \lim_{r \to 0^+} I(p,r).$$

(c) Determine the values of $p > 1$ such that $\ell(p) = 0$.

(a) The infimum is $\geq$ (and actually $=$) than the minimum in $W^{1,p}$, which is not 0 because the function $u \equiv 0$ in $B_1 \setminus B_r$ has not the correct trace in ∂B_r. The existence of the min follows from the direct method

- Notion of convergence: $u_m \to u_\infty$ pointwise a.e. + $\nabla u_m \xrightarrow{L^p} \nabla u_\infty$
- Compactness: if $F(u_m) \leq M$, then $\|\nabla u_m\|_{L^p} \leq M'$ and $\|u_m\|_{L^p} \leq M''$ (BCs in ∂B_r + ineq. à la Poincaré). Then compact embedding $W^{1,p} \to L^p$.
- LSC: LSC of the norm for $\int |\nabla u|^p$. Fatou's lemma for $\int u^{2020}$.

<u>Remark</u> Let us mention some alternative approaches.

- Direct method in $L^q(B_1 \setminus B_r)$ with $q := \min\{p, 2020\}$ with notion of convergence $u_m \to u_\infty$ in L^q and $\nabla u_m \rightharpoonup \nabla u_\infty$ in L^q.
  In this way we do not need Poincaré inequality.

- Trace inequalities of the form

$$\underbrace{\int_{\partial B_r} |u|^q}_{\text{bounded from below because } u \equiv 1 \text{ on } \partial B_r} \leq \int_{\partial (B_1 \setminus B_r)} |u|^q \leq c(q,r) \left\{ \int_{B_1 \setminus B_r} |\nabla u|^q + \int_{B_1 \setminus B_r} |u|^q \right\}$$

with a suitable value of q, for example $q := \min\{p, 2020\}$. Then with some care we can use the RHS in order to provide an estimate from below for $\int_{B_1 \setminus B_r} |\nabla u|^p$ and $\int_{B_1 \setminus B_r} u^{2020}$. Details are left to the reader.

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(b) Monotonicity: Let us observe that

$$I(p,r) = \underbrace{\inf\left\{ \int_{B_1} (|\nabla u|^p + u^{2020}) : \text{same constraints} \right\}}_{J(p,r)} - \underbrace{\text{meas}(B_r)}_{\substack{\downarrow \\ 0 \text{ as } r\to 0}}$$

Now observe that, if $0 < r_1 < r_2$, then $J(p,r_1) \le J(p,r_2)$

(every competitor for $J(p,r_2)$ is a competitor for $J(p,r_1)$)

Remark The monotonicity of $I(p,r)$ is not so clean: as r decreases, the number of competitors increases, but also the integration region increases.

(c) $\ell(p) = 0 \iff \boxed{p \le 3}$ (namely when $W^{1,p} \not\hookrightarrow L^\infty$)

- If $p > 3$ then there exists

$$\min\left\{ \int_{B_1} |\nabla u|^p + u^{2020} : u \in W^{1,p} \text{ and } u(0) \ge 1 \right\} \boxed{>0}$$

crucial point

well defined when $p>3$

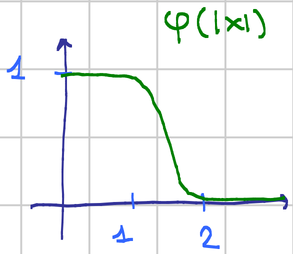
If $r_m \to 0$ and $F(u_m) \le I(p,r_m) + \frac{1}{m}$, then (up to subsequences)

$\to u_m \to u_\infty$ (even uniformly, because $\{u_m\}$ is equi-Hölder cont.)

$\to \liminf F(u_m) \ge F(u_\infty)$

$\to u_\infty$ is a competitor for the min. pblm. above

- If $p<3$, then set $u_r(x) = \varphi\left(\frac{|x|}{r}\right)$ and observe that



$$I(p,r) \le \int_{B_1 \setminus B_r} \underbrace{|\nabla u_r|^p}_{\sim c\cdot \frac{1}{r^p}\cdot r^3} + \underbrace{u_r^{2020}}_{\sim c r^3} \to 0$$

provides one more r when computing $|\nabla u|$

- If $p=3$, then set $u_r(x) = \left[\varphi\left(\frac{|x|}{r}\right)\right]^r$ or better $u_r(x) = \left[\varphi\left(\frac{|x|}{r}\right) + r|x|^2\right]^r$

and observe again that $I(p,r) \to 0$.

this prevents pbms due to the r-th power

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4. For every measurable function $f : [0,1] \to \mathbb{R}$, let us define $Tf : [0,1] \to \mathbb{R}$ as

$$[Tf](x) = \int_0^x t f(t)\, dt \qquad \forall x \in [0,1].$$

Determine whether the restriction of T defines

(a) a bounded operator $C^0([0,1]) \to C^0([0,1])$ (in case, compute the norm of the operator),
(b) a bounded operator $L^2((0,1)) \to L^\infty((0,1))$ (in case, compute the norm of the operator),
(c) a compact operator $L^2((0,1)) \to C^0([0,1])$
(d) an open mapping $L^2((0,1)) \to L^{2020}((0,1))$.

(a) $\boxed{\text{YES}}$ Indeed $\quad |[Tf](x)| \leq \int_0^x t\,|f(t)|\,dt \leq \|f\|_{C^0} \int_0^1 t\,dx \leq \frac{1}{2}\|f\|_{C^0}$

This proves that

$\|Tf\|_{C^0} \leq \frac{1}{2}\|f\|_{C^0}$ with equality for example if $f(x) \equiv 1$.

Therefore the operator is bounded with norm $\boxed{1/2}$

(b) $\boxed{\text{YES}}$ with norm $\boxed{\frac{1}{\sqrt{3}}}$. Indeed

$$|[Tf](x)| \leq \int_0^x t \cdot |f(t)|\,dt \leq \int_0^1 t \cdot |f(t)|\,dt \leq \underbrace{\left\{\int_0^1 t^2\,dt\right\}^{1/2}}_{= \frac{1}{\sqrt{3}}} \underbrace{\left\{\int_0^1 f(t)^2\,dt\right\}^{1/2}}_{= \|f\|_{L^2}}$$

This proves that $\|Tf\|_{L^\infty} \leq \frac{1}{\sqrt{3}}\|f\|_{L^2}$, with equality for ex. if $f(x) = x$.

(c) $\boxed{\text{YES}}$ For every $f \in L^2$ it turns out that $Tf \in H^1$ with $[Tf](0) = 0$

and $[Tf]'(x) = x f(x)$, so that $\|(Tf)'\|_{L^2} \leq \|f\|_{L^2}$.

weak derivative

Therefore, if $\{f_n\} \subseteq L^2$ is bounded, then $\{Tf_n\}$ satisfies the assumptions of Ascoli - Arzelà theorem, and hence it is relatively cpt. in $C^0([0,1])$.

(d) $\boxed{\text{NO}}$ A linear continuous operator between Banach spaces is an open mapping if and only if it is surjective, and T is not surjective because Tf is a continuous function for every $f \in L^2$.

The continuity follows from the continuity of $T : L^2 \to C^0$, which again follows from Hölder inequality.

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