Università di Pisa – Corso di Laurea in Matematica

# Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 21 February 2020

1. Prove that there exists a sequence of functions $u_n : [0,1] \to \mathbb{R}$ of class $C^\infty$ such that
   - $\{u_n\}$ is an orthonormal basis of $L^2((0,1))$,
   - $u_n(0) = u_n(1) = 0$ for every positive integer $n$,
   - for every positive integer $n$, there exists a negative real number $\lambda_n$ such that
   $$(\cos x \cdot u_n'(x))' = \lambda_n u_n(x) \qquad \forall x \in [0,1].$$

2. Discuss existence, uniqueness and regularity of solutions to the boundary value problem
$$u'' = -1 + \sqrt{u}, \qquad u(0) = 1/2, \quad \dot{u}(2020) = 1.$$

3. For every positive real number $R$, let $B_R$ denote the ball in $\mathbb{R}^3$ with center in the origin and radius $R$. For every real number $p > 1$, and every real number $r \in (0,1)$, let us set
$$I(p,r) := \inf \left\{ \int_{B_1 \setminus B_r} \left( |\nabla u|^p + u^{2020} \right) dx : u \in C^\infty(B_1),\ u(x) = 1 \text{ for every } x \in B_r \right\}.$$
   (a) Prove that $I(p,r) > 0$ for every $p > 1$ and every $r \in (0,1)$,
   (b) Prove that for every $p > 1$ there exists
   $$\ell(p) := \lim_{r \to 0^+} I(p,r).$$
   (c) Determine the values of $p > 1$ such that $\ell(p) = 0$.

4. For every measurable function $f : [0,1] \to \mathbb{R}$, let us define $Tf : [0,1] \to \mathbb{R}$ as
$$[Tf](x) = \int_0^x t f(t)\, dt \qquad \forall x \in [0,1].$$
   Determine whether the restriction of $T$ defines
   (a) a bounded operator $C^0([0,1]) \to C^0([0,1])$ (in case, compute the norm of the operator),
   (b) a bounded operator $L^2((0,1)) \to L^\infty((0,1))$ (in case, compute the norm of the operator),
   (c) a compact operator $L^2((0,1)) \to C^0([0,1])$
   (d) an open mapping $L^2((0,1)) \to L^{2020}((0,1))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.