Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 31 January 2020

1. Let us consider the functionals

$$F(u) = u(0) + \int_0^1 \left(\dot{u}^2 + u^2\right)\,dx, \qquad G(u) = [u(0)]^3 + \int_0^1 \left(\dot{u}^2 + u^2\right)\,dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary condition $u(1) = 3$.
   (b) Discuss the minimum problem for $G(u)$ with boundary condition $u(1) = 3$.

2. Let a be a positive real number, and let us consider the boundary value problem

$$u'' = \log u, \qquad u(0) = u(2020) = a.$$

   (a) Discuss existence, uniqueness and regularity of solutions.
   (b) Determine the values of a for which solutions are less than 1 for every $x \in [0, 2020]$.

3. Let B denote an open ball in $\mathbb{R}^3$. For every real number $p > 1$, let us set

$$S(p) := \sup\left\{\int_B u^5\,dx : u \in C^\infty(B),\ \int_B |\nabla u|^p\,dx = \int_B u\,dx = 5\right\}.$$

   (a) Determine whether there exists $p_0 < 2$ such that $S(p)$ is a real number for every $p \geq p_0$.
   (b) Determine whether there exists a real number $p > 1$ such that $S(p) = +\infty$.
   (c) Determine whether there exists a real number M such that $S(p) \leq M$ for every $p \geq 2$.

4. For every sequence $\{x_n\}$ of real numbers, let us set

$$C(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots\right).$$

   In other words, $C(\{x_n\})$ is the sequence $\{y_n\}$ with

$$y_n := \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \forall n \geq 1.$$

   Determine whether the restriction of C defines

   (a) a bounded operator $\ell^1 \to \ell^1$,
   (b) a bounded operator $\ell^1 \to \ell^2$,
   (c) a bounded operator $c \to c$ (as usual c denotes the space of sequences with a finite limit),
   (d) a compact operator $\ell^\infty \to \ell^\infty$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.