Note Title 10/01/2020

1. Determine for which values of the real parameter a the problem

$$\min\left\{\int_{-\pi}^{\pi}\{(\dot{u}-\cos x)^2+(u-\sin x)^2\}\,dx : u\in C^1([-\pi,\pi]),\ u(0)=a\right\}$$

admits a solution (note that the condition is given in the midpoint of the interval).

2. Discuss existence, uniqueness, and regularity of functions $u:\mathbb{R}\to\mathbb{R}$ that are *periodic* and satisfy

$$u''=u^3+\sin^2 x \qquad \forall x\in\mathbb{R}.$$

See CdV 2020-1, Problems 1 and 2

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3. For every positive real numbers R, c, and α, let us set

$$I(R,c,\alpha) := \inf\left\{\int_{B_R} (|\nabla u|^2 - cu^2)\,dx : u \in C^\infty(B_R)\cap H^1(B_R),\ \int_{B_R} u\,dx = \alpha\right\},$$

where B_R denotes the ball in $\mathbb{R}^3$ with center in the origin and radius R.

(a) Determine whether there exists $c > 0$ such that $I(1,c,0) = 0$.
(b) Determine whether there exists $c > 0$ such that $I(1,c,0) = -\infty$.
(c) Determine whether there exists $R > 0$ such that $I(R,1,2020) = -\infty$.

(a) $\boxed{\text{YES}}$ More precisely, this is true when c is small enough. Indeed, from Wirtinger inequality we know that

$$\int_{B_1} u^2 \le k\int_{B_1} |\nabla u|^2 \qquad \text{if } \int_{B_1} u = 0$$

This implies that $I(1,c,0) = 0$ whenever $ck \le 1$

(b) $\boxed{\text{YES}}$ More precisely, this is true when c is large enough. Indeed, let $\varphi \in C^\infty_c(B_1)$ be any function with $\int_{B_1}\varphi = 0$ and $\varphi \not\equiv 0$. Then when c is large enough it turns out that

$$\int_{B_1} |\nabla\varphi|^2 - c\varphi^2 < 0$$

Now consider $u(x) := n\varphi(x)$, and let $n \to +\infty$.

(c) $\boxed{\text{YES}}$ More precisely, this is true when R is large enough. Indeed, let us consider φ as in point (b), and let us set

$$u_R(x) := \varphi\left(\frac{x}{R}\right) + \frac{2020}{\text{meas}(B_R)}$$

Then simple variable changes show that

$$\int_{B_R} |\nabla u_R(x)|^2 = \int_{B_R} \frac{1}{R^2}\left|\nabla\varphi\left(\frac{x}{R}\right)\right| = R\int_{B_1} |\nabla\varphi(y)|^2 dy$$

$$\int_{B_R} |u_R(x)|^2 = \int_{B_R}\left[\varphi\left(\frac{x}{R}\right) + \frac{2020}{\text{meas}(B_R)}\right]^2 = R^3\int_{B_1}\varphi(y)^2 dy + o(R^3)$$

from which the conclusion follows.

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The very same argument shows that the constant in Wirtinger inequality in B_R scales as R^2 in any space dimension.

4. For every measurable function $f : [0,1] \to \mathbb{R}$, let us set

$$[Tf](x) = \int_0^{\sin x} \sin(f(t))\,dt \qquad \forall x \in [0,1].$$

Determine whether the restriction of T defines

(a) a continuous mapping $L^2((0,1)) \to L^{2020}((0,1))$,

(b) a compact mapping $L^{2020}((0,1)) \to L^2((0,1))$,

(c) a compact mapping $C^0([0,1]) \to C^1([0,1])$.

(a) $\boxed{\text{YES}}$ The shortest proof probably consists in showing that $T : L^2((0,1)) \to L^\infty((0,1))$ is Lipschitz continuous, and then exploiting the embedding $L^\infty((0,1)) \to L^{2020}((0,1))$.

The Lipschitz continuity follows from the estimate

$$|[Tf](x) - [Tg](x)| = \left| \int_0^{\sin x} (\sin f(t) - \sin g(t))\,dt \right|$$

$$\le \int_0^{\sin x} |\sin f(t) - \sin g(t)|\,dt \underset{\substack{\uparrow \\ \sin x \text{ is Lip} \\ \sin x \le 1}}{\le} \int_0^1 |f(t) - g(t)| \underset{\substack{\uparrow \\ \text{Hölder}}}{\le} \left\{ \int_0^1 |f(t) - g(t)|^2 dt \right\}^{1/2}$$

$$= \|f - g\|_{L^2}$$

(b) $\boxed{\text{YES}}$ If $\{f_n\} \subseteq L^{2020}((0,1))$ is any sequence, then $\{Tf_n\}$ is a sequence of bounded functions with bounded derivative. Due to Ascoli–Arzelà theorem, the sequence $\{Tf_n\}$ is relatively compact in $C^0([0,1])$, and hence a fortiori in $L^2((0,1))$.

(c) $\boxed{\text{NO}}$ Let us consider the sequence $f_n(x) := (1-x)^n$. This sequence is bounded in $C^0([0,1])$. We claim that $\{[Tf_n]'(x)\}$, namely the sequence of derivatives of the images, has no converging subsequence, and therefore $\{Tf_n\}$ is not bounded in $C^1([0,1])$. Indeed

$$[Tf_n]'(x) = \sin(f_n(\sin x)) \cdot \cos x = \sin((1 - \sin x)^n) \cos x$$

The pointwise limit of this sequence is $\sin(1)$ if $x = 0$, and 0 if $x \in (0,1]$. Since the pointwise limit is not continuous, the sequence does NOT admit any converging subsequence.