Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 11 January 2020

1. Determine for which values of the real parameter a the problem

$$\min\left\{\int_{-\pi}^{\pi}\left\{(\dot{u}-\cos x)^2+(u-\sin x)^2\right\}\,dx : u\in C^1([-\pi,\pi]),\ u(0)=a\right\}$$

   admits a solution (note that the condition is given in the midpoint of the interval).

2. Discuss existence, uniqueness, and regularity of functions $u:\mathbb{R}\to\mathbb{R}$ that are *periodic* and satisfy

$$u''=u^3+\sin^2 x \qquad \forall x\in\mathbb{R}.$$

3. For every positive real numbers R, c, and α, let us set

$$I(R,c,\alpha):=\inf\left\{\int_{B_R}\left(|\nabla u|^2-cu^2\right)\,dx : u\in C^\infty(B_R)\cap H^1(B_R),\ \int_{B_R}u\,dx=\alpha\right\},$$

   where B_R denotes the open ball in $\mathbb{R}^3$ with center in the origin and radius R.

   (a) Determine whether there exists $c>0$ such that $I(1,c,0)=0$.

   (b) Determine whether there exists $c>0$ such that $I(1,c,0)=-\infty$.

   (c) Determine whether there exists $R>0$ such that $I(R,1,2020)=-\infty$.

4. For every measurable function $f:[0,1]\to\mathbb{R}$, let us set

$$[Tf](x)=\int_0^{\sin x}\sin(f(t))\,dt \qquad \forall x\in[0,1].$$

   Determine whether the restriction of T defines

   (a) a continuous mapping $L^2((0,1))\to L^{2020}((0,1))$,

   (b) a compact mapping $L^{2020}((0,1))\to L^2((0,1))$,

   (c) a compact mapping $C^0([0,1])\to C^1([0,1])$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.