Università di Pisa – Corso di Laurea in Matematica

Exam paper of "Istituzioni di Analisi Matematica"

Pisa, 03 September 2019

1. Determine whether the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 + \dot{u}u + u^2 + u\right)\, dx$$

has the minimum in the class $C^1([0,1])$.

2. For every positive integer d, let us consider the following three inequalities:

$$\int_{\mathbb{R}^d} u(x)^{32}\, dx \leq K_d, \qquad u(0) \leq K_d, \qquad \|\nabla u(0)\| \leq K_d,$$

For each of them, determine the values of d for which there exists a constant K_d that makes it true for every $u \in C^\infty_c(\mathbb{R}^d)$ whose norm in $W^{20,19}(\mathbb{R}^d)$ is less than or equal to 1.

3. Let us consider the square $\Omega = (0,1)^2$, and for every real number $\varepsilon > 0$ let us set

$$I(\varepsilon) := \inf\left\{\int_\Omega \left(u_x^2 + u_y^2 + u^7\right)\, dx\, dy : u \in C^1_c(\Omega),\ \int_\Omega \left(u_x^2 + 5u_y^2\right)\, dx\, dy \leq \varepsilon\right\}.$$

   (a) Prove that $I(\varepsilon)$ is a real number for every $\varepsilon > 0$.

   (b) Determine whether there exists $\varepsilon > 0$ such that $I(\varepsilon) = 0$.

   (c) Find the limit of $I(\varepsilon)$ as $\varepsilon \to +\infty$.

4. For every $f : (0,1) \to \mathbb{R}$, let us set

$$[Tf](x) := f(x^2) \qquad \forall x \in (0,1).$$

Determine whether the restriction of T defines

   (a) a *continuous* operator $H^1((0,1)) \to L^4((0,1))$,

   (b) a *continuous* operator $W^{1,4}((0,1)) \to W^{1,4}((0,1))$,

   (c) a *compact* operator $H^1((0,1)) \to H^1((0,1))$.

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.