

Proof of open mapping thm (1) $\Rightarrow$ (2) easy because
 open $\Rightarrow$ (quantitative) solver $\Rightarrow$ surjective

(2) $\Rightarrow$ (1) We prove that (3) of Prop 2 is satisfied, namely
 $\exists R > 0$ s.t. $\text{Clos}_W(\varphi(B_V(0, R))) \supseteq B_W(0, 1)$.

Here we use that W is Banach. Consider

$$C_k := \text{Clos}_W(\varphi(B_V(0, k)))$$

Since φ is surjective, then $\bigcup_{k=1}^{\infty} C_k = W$

Baire category $\Rightarrow \exists k_0$ s.t. $B_W(w_0, r_0) \subseteq C_{k_0}$

Claim: $B_W(0, 1) \subseteq C_{\frac{2k_0}{r_0}} = R$

Take any $w \in B_W(0, 1)$ and write

$$w = \frac{1}{r_0} \underbrace{(w_0 + r_0 w)}_{B_W(w_0, r_0)} - \frac{1}{r_0} \underbrace{w_0}_{B_W(w_0, r_0)}$$

By assumption $\exists \varphi(u_m) \rightarrow w_0 + r_0 w$ $u_m \in B_V(0, k_0)$
 $\exists \varphi(\hat{u}_m) \rightarrow w_0$ $\hat{u}_m \in B_V(0, k_0)$

Now observe

$$\underbrace{\varphi\left(\frac{1}{r_0} u_m - \frac{1}{r_0} \hat{u}_m\right)}_{\in B_V(0, \frac{2k_0}{r_0})} \rightarrow w$$

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Prop 2 (1) $\Rightarrow$ (2) Trivial

(2) $\Rightarrow$ (3) Let $\hat{D}$ the set where the quantitative solver is defined. Let $w \in B_W(0,1)$ and let $w_n \xrightarrow{n \in \hat{D}} w$

For n large enough $\|w_n\| < 1$. Consider $v_n := \hat{S}(w_n)$

By definition $\varphi(v_n) = w_n \rightarrow w$ and

$$\|v_n\|_V = \|\hat{S}(w_n)\|_V \leq \hat{M} \|w_n\|_W \leq \hat{M} = R$$

Therefore $\text{Clos}(\varphi(B_V(0,R))) \supseteq B_W(0,1)$.

(3) $\Rightarrow$ (2) Observe that $\text{Clos}(\varphi(B_V(0,R))) \supseteq \bar{B}_W(0,1)$.

Observe also that (3) implies that the image of φ is dense in W . Let $\hat{D}$ denote the image.

Now we define a solver in $\hat{D}$. Let $w \in \hat{D}$. Then

$$\frac{w}{\|w\|_W} \in \text{Image of } \varphi \cap \bar{B}_W(0,1)$$

$\Rightarrow \exists v \in B_V(0,R)$ s.t. $\varphi(v) = \frac{w}{\|w\|_W}$. Now define

$\hat{S}(w) := \|w\|_W \cdot v$ and observe that it works with

$$\|\hat{S}(w)\|_V = \underbrace{\|v\|_V}_{\leq R} \cdot \|w\|_W$$

(2) $\Rightarrow$ (1) relies on a lemma (UP TO NOW WE DID NOT USE THAT V IS BANACH AND THAT φ IS CONTINUOUS)

[Added after video: NORMED is enough.]

Lemma Let W be a Banach space, and let $D \subseteq W$ be a dense subset. Then for every $w \in W$ there exists $\{w_i\} \subseteq D$ such that

$$\sum_{i=1}^{\infty} w_i = w \quad \text{and} \quad \sum_{i=1}^{\infty} \|w_i\|_W \leq 2 \|w\|_W$$

Proof We can replace 2 in the last inequality with any constant > 1 .

Proof Choose $w_1 \in \mathcal{D}$ s.t. $\|w - w_1\|_w \leq \frac{1}{8} \|w\|_w$

Observe that

$$\|w_1\|_w \leq \|w\|_w + \|w_1 - w\|_w \leq \|w\|_w + \frac{1}{2} \|w\|_w$$

$\uparrow$ generous

Choose $w_2 \in \mathcal{D}$ s.t. $\|w - w_1 - w_2\|_w \leq \frac{1}{16} \|w\|_w$

Observe that

$$\begin{aligned} \|w_2\|_w &\leq \|w - w_1\|_w + \|w_2 + w_1 - w\|_w \leq \frac{1}{4} \|w\|_w \\ &\leq \frac{1}{8} \|w\|_w \quad \leq \frac{1}{16} \|w\|_w \end{aligned}$$

... and so on ... at step k we choose $w_k \in \mathcal{D}$ s.t.

$$\|w - w_1 - \dots - w_k\|_w \leq \frac{1}{2^{k+2}} \|w\|_w \quad (*)$$

and observe that

$$\begin{aligned} \|w_k\|_w &\leq \underbrace{\|w - w_1 - \dots - w_{k-1}\|_w}_{\leq \frac{1}{2^{k+1}} \|w\|_w} + \underbrace{\|w_k + w_{k-1} + \dots + w_1 - w\|_w}_{\leq \frac{1}{2^{k+2}} \|w\|_w} \leq \frac{1}{2^k} \|w\|_w \end{aligned}$$

Observe that $(*)$ implies that $w = \sum_{i=1}^{\infty} w_i$ and

$$\begin{aligned} \sum_{i=1}^{\infty} \|w_i\|_w &= \|w_1\|_w + \sum_{i=2}^{\infty} \|w_i\|_w \\ &\leq \|w\|_w + \frac{1}{2} \|w\|_w + \underbrace{\|w\|_w \sum_{i=2}^{\infty} \frac{1}{2^i}}_{\|w\|_w} \end{aligned}$$

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Proof of (2) $\Rightarrow$ (1) Let $D \subseteq W$ be the dense subset where the quantitative solver is defined.

Goal: extend S to the whole space W .

Take any $w \in W$. Consider $\{w_m\} \subseteq D$ as in the lemma.

Consider $v_m := \hat{S}(w_m)$ and define

$$v := \sum_{m=1}^{\infty} \hat{S}(w_m)$$

Is it well defined? Yes, because

$$\sum_{n=1}^{\infty} \|\hat{S}(w_m)\|_V \leq \hat{M} \sum_{n=1}^{\infty} \|w_m\|_W \leq \hat{M} \cdot 2 \|w\|_W$$

(Absolute conv. $\Rightarrow$ conv. because V is Banach)

$$f(v) = f\left(\sum_{n=1}^{\infty} \hat{S}(w_m)\right) = \sum_{n=1}^{\infty} f(\hat{S}(w_m)) = \sum_{n=1}^{\infty} w_m = w.$$

$\uparrow$
 f IS CONTINUOUS
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ANSWER TO Q4

Theorem Assume $f: H \rightarrow W$ is linear and continuous
 $\uparrow \quad \quad \uparrow$
HILBERT BANACH

Assume f is surjective.

Then a quantitative linear solver exists (and therefore also continuous)

Proof Let $K := \ker(f)$. Observe that $H = K \oplus K^\perp$
 $\uparrow$
orthogonal

Simple facts: $K^\perp$ is closed (and therefore a Hilbert space)

$f: K^\perp \rightarrow W$ is surjective and injective

Given $w \in W$ there exists $v \in H$ s.t. $f(v) = w$. But

$v = v_1 + v_2$ and observe that
 $\overset{\text{in}}{\underset{K}{K}} \quad \overset{\text{in}}{\underset{K^\perp}{K^\perp}}$

$$w = f(v) = f(v_1) + f(v_2) = f(v_2)$$

$\underset{0}{\parallel}$

Therefore $f: K^\perp \rightarrow W$ is invertible and linear and cont. between Banach spaces.

The inverse is the required solver.

Remark The same proof works for $f: V \rightarrow W$
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BANACH

provided that $K := \ker(f)$ has a TOPOLOGICAL COMPLEMENT in V , namely

$$V = K \oplus M$$

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CLOSED vector subspace

BAD NEWS If every closed subspace $K \subseteq V$ has a top. complement, then the norm of V is equivalent to a norm generated by a scalar product.

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[CREDITS TO TERENCE TAO FOR THESE TWO LECTURES].