

Given $L \in (L^p)'$ we found $g \in L^1$ such that

$$L(1_A) = \int_A g(x) d\mu = [J(g)](1_A)$$

Claim: $g \in L^{p'}$ (which is better than L^1 because $\mu(X) < \infty$)

[p=1] we claim $\|g\|_{L^\infty} \leq \|L\|_{(L^1)'}$

Assume this is not the case. Then there exists A_ε with $\text{meas}(A_\varepsilon) > 0$ s.t.

$$|g(x)| \geq \|L\|_{(L^1)'} + \varepsilon \quad \forall x \in A_\varepsilon$$

WLOG we can assume the same with $g(x)$ in the LHS instead of $|g(x)|$. Now we observe that

$$\begin{aligned} (\|L\|_{(L^1)'} + \varepsilon) \text{meas}(A_\varepsilon) &\leq \int_{A_\varepsilon} g(x) d\mu \\ &= [J(g)](1_{A_\varepsilon}) \\ &= L(1_{A_\varepsilon}) \\ &\leq \|L\|_{(L^1)'} \cdot \|1_{A_\varepsilon}\|_{L^1} \\ &= \|L\|_{(L^1)'} \cdot \text{meas}(A_\varepsilon) \end{aligned}$$

If we simplify, we obtain a contradiction.

$p \in (1, +\infty)$ We know that L and $J(g)$ coincide on $\mathbb{1}_A$ and $g \in L^1$.

This implies that they coincide in $L^\infty(X)$
(here we use that the span of $\mathbb{1}_A$ is dense in $L^\infty(X)$)

Given $g(x)$ let us consider a truncation

$$g_n(x) := \begin{cases} n & \text{if } g(x) \geq n \\ g(x) & \text{if } -n \leq g(x) \leq n \\ -n & \text{if } g(x) \leq -n \end{cases}$$

and let us set $f_n(x) := |g_n(x)|^{p-1} \cdot \underbrace{\text{sign}(g_n(x))}_{\text{sign}(g(x))} \in L^\infty$

$$\begin{aligned} \int_X |g_n(x)|^{p'} d\mu &= \int_X |g_n(x)|^{p'-1} \text{sign}(g_n(x)) \cdot g_n(x) d\mu \\ &= \int_X |g_n(x)|^{p'-1} \underbrace{\text{sign}(g(x)) g_n(x)}_{\text{sign}(g(x)) \cdot g(x)} d\mu \end{aligned}$$

$$\leq \int_X f_n(x) g(x) d\mu$$

$$= [J(g)](f_n) \quad \text{since } J(g) = L \text{ on } L^\infty$$

$$= L(f_n)$$

$$\leq \|L\|_{(L^p)'} \|f_n\|_{L^p}$$

$$= \|L\|_{(L^p)'} \left\{ \int_X |g_n(x)|^{p'} d\mu \right\}^{1/p}$$

If we simplify we obtain that

$$\|g_n\|_{L^{p'}} \leq \|L\|_{(L^p)'}.$$

We let $n \rightarrow +\infty$ and obtain that $\|g\|_{L^{p'}} \leq \|L\|_{(L^p)'}$.
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Passing from $\mu(X) < +\infty$ and σ -finite measure, namely X is the increasing union of subsets X_k with $\mu(X_k) < +\infty$.

Step 1 Consider $L: L^p(X) \rightarrow \mathbb{R}$ linear and continuous

By restriction it defines

$$L_k: L^p(X_k) \rightarrow \mathbb{R}$$

By the previous result

$$L_k(f) = \int_{X_k} g_k(x) f(x) d\mu \quad \forall f \in L^p(X_k)$$

Step 2 If $X_k \subseteq X_l$, then $g_k(x) = g_l(x)$ for every $x \in X_k$

(it is some kind of FLCV)

and therefore we can define

$$g(x) := g_k(x) \quad \text{if } x \in X_k$$

Step 3 We observe that

$$\begin{aligned} \|g\|_{L^{p'}(X_k)} &= \|g_k\|_{L^{p'}(X_k)} = \|L_k\|_{(L^p(X_k))'} \\ &\leq \|L\|_{(L^p(X))'} \end{aligned}$$

and this implies that $g \in L^{p'}(X)$. We conclude that

$J(g)$ and L coincide on $L^p(X)$.
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HILBERT SPACES

Short version : $(H)' = H$

Duality pairing $[J(v)](w) := \langle v, w \rangle \quad \forall (v, w) \in H^2$

Theorem The duality pairing defines a map

$$H \ni v \longrightarrow J(v) \in H'$$

which has the usual properties

→ well-defined, linear, continuous

→ isometry

→ surjective.

Proof of first two parts

$$|[J(v)](w)| = |\langle v, w \rangle| \leq \|v\|_H \cdot \|w\|_H$$

This proves that $\|J(v)\|_{H'} \leq \|v\|_H$

In order to prove the opposite inequality we choose $w := v$ and observe that

$$\begin{aligned} \|v\|_H^2 &= \langle v, w \rangle = [J(v)](w) \leq \|J(v)\|_{H'} \cdot \|w\|_H = \\ &= \|J(v)\|_{H'} \cdot \|v\|_H \end{aligned}$$

If we simplify we conclude (consider also $v=0$)

Proof #1 of surjectivity

Assume that H is separable and let $\{e_i\}$ be a Hilbert basis

(if H is not separable the same argument works with a general Hilbert basis $\{e_i\}_{i \in I}$)

Let $L: H \rightarrow \mathbb{R}$ be linear and continuous.

We need to find $v \in H$ such that

$$L(w) = \langle v, w \rangle \quad \forall w \in H$$

Let us set

$$v := \sum_{k=1}^{\infty} \underbrace{L(e_k)}_{v_k} e_k$$

Assume that v is well-defined. Then $L(w)$ and $\langle v, w \rangle$ coincide in e_k for every $k \geq 1$, and therefore in H .

For every $n \geq 1$ observe that

$$\begin{aligned} \sum_{k=1}^n v_k^2 &= \sum_{k=1}^n v_k L(e_k) = L\left(\sum_{k=1}^n v_k e_k\right) \\ &\leq \|L\|_{H'} \cdot \left\| \sum_{k=1}^n v_k e_k \right\| \\ &= \|L\|_{H'} \cdot \left\{ \sum_{k=1}^n v_k^2 \right\}^{1/2} \end{aligned}$$

We simplify and we let $n \rightarrow \infty$.

Proof #2 of surjectivity let $K := \text{Ker}(L)$. If $K = H$, then $v=0$ represents L .

Assume that $K \neq H$. let $v_0 \neq 0$ be any vector in $K^\perp$ (here we use that $H = K \oplus K^\perp$)

Every $w \in H$ can be written as $w = \alpha v_0 + w_\perp$ with $\alpha \in \mathbb{R}$ and $w_\perp \in \text{Ker}(L)$. What is α ?

$$\langle w, v_0 \rangle = \alpha \|v_0\|^2 + \underbrace{\langle w_\perp, v_0 \rangle}_0 \quad \leadsto \quad \alpha = \frac{\langle v_0, w \rangle}{\|v_0\|^2}$$

Observe that

$$L(w) = L(\alpha v_0 + w_\perp) = \alpha L(v_0) + L(w_\perp) \\ = 0$$

$$= \alpha L(v_0)$$

$$= \frac{\langle v_0, w \rangle}{\|v_0\|^2} L(v_0)$$

$$= \left\langle \frac{L(v_0)}{\|v_0\|^2} v_0, w \right\rangle$$

↑
vector v that represents L .
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[Added after video: when we write $w = \alpha v_0 + w_\perp$ we need to know that $K^\perp = \text{span}(v_0)$, namely that $K^\perp$ has dimension 1. This is true because, if v_1 and v_2 are vectors in $K^\perp$, then $L(v_1)v_1 - L(v_2)v_2 \in K^\perp \cap K$, and hence it is 0, and therefore v_1 and v_2 are linearly dependent.

Alternative proof: it turns out that $w = \alpha v_0 + w_\perp$ if we choose $\alpha := \frac{L(w)}{L(v_0)}$ (just check that the difference $w - \alpha v_0 \in \text{Ker}$)