

Setting: X and Y are NORMED SPACES (almost everything works up to locally convex topological vector spaces)

Questions: ① $\exists f: X \rightarrow Y$ linear and $f \neq 0$
 ② $\exists f: X \rightarrow Y$ linear and CONTINUOUS and $f \neq 0$
 ③ $\exists f: X \rightarrow Y$ linear and NON-CONTINUOUS

Answer ① YES, but we need CHOICE AXIOM (CA)

Proof CA $\Rightarrow$ existence of HAMEL BASIS in X (algebraic basis)
 $\uparrow$ every $x \in X$ is FINITE linear combination

Let $\{x_i\}_{i \in I}$ be such a basis. Define $f(x_i) = y_i$
 $\uparrow$ elements of Y
 and extend by linearity

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PROP ($\neq$ shades of continuity for LINEAR MAPS)

Let $F: X \rightarrow Y$ be a linear map.

Then the following are equivalent.

- (i) f is ε/δ continuous
- (ii) f is sequentially continuous
- (iii) f (every ball in X) is bounded in Y
- (i') f is ε/δ continuous in $x=0$
- (ii') f is seq. cont. in $x=0$
- (iii') $f(B_X(0,1))$ is bounded in Y
- (vii) f is LIPSCHITZ CONTINUOUS

Proof Trivial : (i) $\Rightarrow$ (i'), (ii) $\Rightarrow$ (ii'), (iii) $\Rightarrow$ (iii')

(vii) $\Rightarrow$ all the rest

Standard : (i) $\Leftrightarrow$ (ii) (general fact in metric spaces)

(i') $\Rightarrow$ (iii') Apply ε/δ cont. in $x=0$ with $\varepsilon = 1$.

We find $\delta > 0$ such that $f(B_X(0, \delta)) \subseteq B_Y(0, 1)$

By linearity

$$f(B_X(0, 1)) = f\left(\frac{1}{\delta} B_X(0, \delta)\right) \subseteq \frac{1}{\delta} B_Y(0, 1)$$

$\uparrow$
linearity

(i) $\Rightarrow$ (vii) As above there exists $\delta > 0$ such that

$$f(\bar{B}_X(0, 1)) \subseteq \frac{1}{\delta} \bar{B}_Y(0, 1)$$

For every $x \in X$ with $x \neq 0$ consider $\frac{x}{\|x\|_X} \in \bar{B}_X(0, 1)$
then

$$f(x) = \|x\|_X \underset{\substack{\uparrow \\ \text{linearity}}}{f}\left(\frac{x}{\|x\|_X}\right)$$

and therefore

$$\|f(x)\|_Y = \|x\|_X \left\| f\left(\frac{x}{\|x\|_X}\right) \right\|_Y \leq \frac{1}{\delta} \|x\|_X.$$

$\in \bar{B}_Y(0, \frac{1}{\delta})$

By linearity

$$\|f(x_2) - f(x_1)\|_Y = \|f(x_2 - x_1)\|_Y \leq \frac{1}{\delta} \|x_2 - x_1\|_X.$$

$\uparrow$
Lip. constant

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Answer ③ YES if $\dim(X) = +\infty$ and we accept C.A.

Proof Consider $Y = \mathbb{R}$. We construct $f: X \rightarrow \mathbb{R}$ linear and NOT continuous.

Let $\{x_n\} \subseteq X$ be linearly indep. vectors (can assume $\|x_n\|_X = 1$)

CA $\Rightarrow$ we can complete them to Hamel basis

Define $f(x_n) = n$ for every $n \geq 1$

f (rest of the basis) = what we want (for ex. 0)

and extend by linearity.

The image of $B_X(0,1)$ is unbounded.
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Def. (PSEUDO-NORM)

Let X be a VECTOR SPACE. A pseudo-norm is a function

$$p: X \rightarrow \mathbb{R}$$

with two properties

(i) (subadditivity)

$$p(x_1 + x_2) \leq p(x_1) + p(x_2) \quad \forall (x_1, x_2) \in X^2$$

(ii) (positively homogeneous)

$$p(\lambda x) = \lambda p(x) \quad \forall x \in X \quad \forall \lambda > 0$$

Remark The one w/o sign conditions

Theorem (HANN-BANACH, analytic form)

Let X be a vector space. Let p be a pseudo-norm on X .

Let $E \subseteq X$ be a vector subspace. Let $f: E \rightarrow \mathbb{R}$ be a linear map such that

$$f(x) \leq p(x) \quad \forall x \in E$$

Then there exists $\hat{f}: X \rightarrow \mathbb{R}$ linear such that

$$\hat{f}(x) = f(x) \quad \forall x \in E$$

$$\hat{f}(x) \leq p(x) \quad \forall x \in X$$

Proof Let $\mathcal{Y}$ denote the set of extensions of f . More precisely an element of $\mathcal{Y}$ is a pair (F, g) such that

$$\begin{array}{ccc} E \subseteq F & \begin{array}{c} \uparrow \\ \text{subspace} \end{array} & \begin{array}{c} \uparrow \\ \text{lin. map on } F \end{array} \\ g(x) = f(x) \quad \forall x \in E & + & g(x) \leq p(x) \quad \forall x \in F \end{array}$$

• $\mathcal{Y}$ is nonempty (it contains (E, f))

• $\mathcal{Y}$ is partially ordered by inclusion $(F_1, g_1) \leq (F_2, g_2)$ if $F_1 \subseteq F_2$ and $g_2(x) = g_1(x) \quad \forall x \in F_1$

• If $(F_i, g_i)_{i \in I}$ is a totally ordered subset of $\mathcal{Y}$
 $\uparrow$
not necessarily countable

then it admits an upper bound in $\mathcal{Y}$ defined as

$$\hat{F} := \bigcup_{i \in I} F_i \quad \hat{g}(x) = g_i(x) \quad \text{if } x \in F_i$$

[check that this is a well defined element of $\mathcal{Y}$]

ZORN LEMMA $\Rightarrow \mathcal{Y}$ admits a maximal element.

Let (F, g) denote this element. Claim: $F = X$.

Assume it is not the case, namely $x_0 \in X \setminus F$.

In this case we extend g to $F \oplus \mathbb{R}x_0$.

Is it possible? We have to define

$$g(u + \alpha x_0) \quad \forall u \in F \quad \forall \alpha \in \mathbb{R} \\ = g(u) + \alpha g(x_0)$$

We need to find $g(x_0)$ in such a way that

$$g(u) + \alpha g(x_0) \leq p(u + \alpha x_0) \quad \forall u \in F \quad \forall \alpha \in \mathbb{R}$$

There are three cases

- $\alpha = 0 \rightsquigarrow$ easy
- $\alpha > 0 \rightsquigarrow g(u + \alpha x_0) \leq p(u + \alpha x_0)$

$$\cancel{g}\left(\frac{u}{\alpha} + x_0\right) \leq \cancel{p}\left(\frac{u}{\alpha} + x_0\right) \\ \uparrow \\ \text{generic element of } F$$

Therefore we need $\boxed{g(y + x_0) \leq p(y + x_0)} \quad \forall y \in F$

- $\alpha < 0 \rightsquigarrow \alpha = -\beta$ with $\beta > 0 \rightsquigarrow g(u - \beta x_0) \leq p(u - \beta x_0)$

$$\cancel{g}\left(\frac{u}{\beta} - x_0\right) \leq \cancel{p}\left(\frac{u}{\beta} - x_0\right)$$

$$\boxed{g(z - x_0) \leq p(z - x_0)} \quad \forall z \in F$$

The two conditions are

$$g(y) + g(x_0) \leq p(y+x_0) \quad \forall y \in F$$

$$g(z) - g(x_0) \leq p(z-x_0) \quad \forall z \in F$$

$$g(z) - p(z-x_0) \leq g(x_0) \leq p(y+x_0) - g(y) \quad \forall (y, z) \in F^2$$

We can find $g(x_0)$ if and only if

$$g(z) - p(z-x_0) \stackrel{?}{\leq} p(y+x_0) - g(y) \quad \forall (y, z) \in F^2$$

namely

$$g(z) + g(y) \stackrel{?}{\leq} p(y+x_0) + p(z-x_0) \quad \forall (y, z) \in F^2$$

$$g(z) + g(y) \stackrel{\substack{\uparrow \\ \text{g linear} \\ \text{on } F}}{=} g(z+y) \stackrel{\substack{\uparrow \\ \text{estimate true in } F}}{\leq} p(z+y) \\ = p(z-x_0 + y+x_0)$$

$$\leq p(z-x_0) + p(y+x_0) \\ \uparrow \\ p \text{ is subadditive}$$

This gives a contradiction, so that $F = \emptyset$.

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