

Sobolev embedding $p < d$

$$\frac{1}{p} - \frac{1}{p_*} = \frac{1}{d}$$

$$u \in W^{1,p}(\mathbb{R}^d) \Rightarrow u \in L^{p_*}(\mathbb{R}^d) \quad \text{where } p_* = \frac{pd}{d-p}$$

Step 1 Assume $u \in C_c^\infty(\mathbb{R}^d)$ and $p=1$ so $p_* = \frac{d}{d-1}$

$$u(x_1, x_2, \dots, x_d) = \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_d) dt$$

↑
negative enough
number

$$|u(x_1, x_2, \dots, x_d)| \leq \underbrace{\int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_d) \right| dt}_{\varphi_1(\hat{x}_1)}$$

In the same way

$$|u(x_1, x_2, \dots, x_d)| \leq \underbrace{\int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d) \right| dt}_{\varphi_i(\hat{x}_i)}$$

If we multiply these relations

$$|u(x)|^d \leq \prod_{i=1}^d \varphi_i(\hat{x}_i)$$

and hence

$$|u(x)|^{\frac{d}{d-1}} \leq \prod_{i=1}^d \varphi_i(\hat{x}_i)^{\frac{1}{d-1}}$$

We are in the assumptions of Gagliardo's ineq.

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx \leq \prod_{i=1}^d \left\{ \int_{\mathbb{R}^{d-1}} \varphi_i(\hat{x}_i) d\hat{x}_i \right\}^{\frac{1}{d-1}}$$

↑
Gagliardo

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \varphi_i(\hat{x}_i) d\hat{x}_i &= \int_{\mathbb{R}^{d-1}} d\hat{x}_i \int_{-\infty}^{+\infty} dt \left| \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d) \right| \\ &= \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial x_i}(x) \right| dx \\ &\leq \int_{\mathbb{R}^d} |\nabla u(x)| dx \end{aligned}$$

If we plug this in the previous estimate we obtain

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx \leq \left\{ \int_{\mathbb{R}^d} |\nabla u(x)| dx \right\}^{\frac{d}{d-1}}$$

This implies that $\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla u\|_{L^1(\mathbb{R}^d)}$

Step 2 Assume that $u \in C_c^\infty(\mathbb{R}^d)$ and $p < d$.

Trick: consider $v(x) := |u(x)|^2 u(x) = \varphi(u(x))$ ($r > 0$)

with $\varphi(\sigma) := |\sigma|^2 \sigma$

Then

$$\varphi'(\sigma) = (2+1)|\sigma|^2 \sigma$$

$$\nabla v(x) = \varphi'(u(x)) \nabla u(x) = (2+1) |u(x)|^2 \nabla u(x).$$

From Step 1 we know that

$$\|v\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|\nabla v\|_{L^1(\mathbb{R}^d)}$$

$$\begin{aligned} \text{LHS} &= \left\{ \int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx \right\}^{\frac{d-1}{d}} = \left\{ \int_{\mathbb{R}^d} |u(x)|^{(n+1)\frac{d}{d-1}} dx \right\}^{\frac{d-1}{d}} \\ &= \|u\|_{L^{(n+1)\frac{d}{d-1}}}^{n+1} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \int_{\mathbb{R}^d} |\nabla u(x)| dx = (n+1) \int_{\mathbb{R}^d} |u(x)|^{\frac{n+1}{p}} \cdot |\nabla u(x)|^{\frac{1}{p}} dx \\ &\leq (n+1) \left\{ \int_{\mathbb{R}^d} |u(x)|^{\frac{np}{p-1}} dx \right\}^{\frac{p-1}{p}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)} \\ &= (n+1) \|u\|_{L^{\frac{np}{p-1}}}^{\frac{np}{p-1}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)} \end{aligned}$$

Putting things together we get

$$\|u\|_{L^{(n+1)\frac{d}{d-1}}}^{n+1} \leq (n+1) \|u\|_{L^{\frac{np}{p-1}}}^{\frac{np}{p-1}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

Idea: choose n in such a way that $(n+1)\frac{d}{d-1} = \frac{np}{p-1}$

$$n \left(\frac{p}{p-1} - \frac{d}{d-1} \right) = \frac{d}{d-1} \quad \leadsto \quad n \frac{\cancel{p}d - \cancel{p} - \cancel{d}p + d}{(p-1)\cancel{(d-1)}} = \frac{d}{\cancel{d-1}}$$

$$\begin{aligned} \leadsto n &= \frac{d(p-1)}{d-p} \quad \leadsto \quad (n+1)\frac{d}{d-1} = \left(\frac{d(p-1)}{d-p} + 1 \right) \frac{d}{d-1} \\ &= \frac{dp - \cancel{d} + \cancel{d} - p}{d-p} \frac{d}{d-1} = \frac{pd}{d-p} = p^* \end{aligned}$$

with this choice of n we obtain

$$\|u\|_{L^{p^*}}^{n+1} \leq \overset{C(p,d)}{\uparrow} (n+1) \|u\|_{L^{\frac{np}{p-1}}}^{\frac{np}{p-1}} \cdot \|\nabla u\|_{L^p} \quad \leadsto \text{conclusion}$$

Step 3 Let us consider now any $u \in W^{1,p}(\mathbb{R}^d)$, and let us approximate it in the deluxe way with $\{u_n\} \in C_c^\infty(\mathbb{R}^d)$

$$u_m \rightarrow u \text{ in } L^p(\mathbb{R}^d) \quad \nabla u_m \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^d)$$

$\{\nabla u_n\}$ is a Cauchy sequence in $L^p(\mathbb{R}^d)$ (H is convergent)

As a consequence, $\{u_n\}$ is a Cauchy sequence in $L^{p_*}(\mathbb{R}^d)$ because

$$\|u_m - u_n\|_{L^{p_\infty}(\mathbb{R}^d)} \leq C(p, d) \left\| \begin{array}{c} \nabla u_m - \nabla u_n \\ \nabla(u_m - u_n) \end{array} \right\|_{L^p(\mathbb{R}^d)}$$

Therefore $u_m \rightarrow \hat{u}$ in $L^{p^*}(\mathbb{R}^d)$
 $u_m \rightarrow u$ in $L^p(\mathbb{R}^d)$

The two limits coincide (L^q convergence $\Rightarrow$ pointwise conv. of a subsequence)

At this point we can pass to the limit in

$$\|u_m\|_{L^{p^*}(\mathbb{R}^d)} \leq C(p, d) \|\nabla u_m\|_{L^p(\mathbb{R}^d)}$$

Sobolev embedding in the case $p=d$

$$u \in W^{1,d}(\mathbb{R}^d) \Rightarrow u \in L^q(\mathbb{R}^d) \quad \forall q \in [p, +\infty) \text{ with some estimate}$$

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C(d, q) \|u\|_{1, d, \mathbb{R}^d}$$

As in the previous case, we obtain that

$$\|u\|_{L^{(r+1)\frac{d}{d-1}}}^{r+1} \leq (r+1) \|u\|_{L^{r\frac{d}{d-1}}}^r \|\nabla u\|_{L^d(\mathbb{R}^d)}$$

Key point: the summability in the LHS is better than in the RHS

Let us set $p_k := d + \frac{d}{d-1}k$ and let us prove by induction that $u \in L^{p_k}(\mathbb{R}^d)$.

This is enough because $p_k \rightarrow +\infty$.

Case $k=0$ Trivial, because $u \in L^d(\mathbb{R}^d)$ by assumption.
Here the estimate is already impure

Case $k \Rightarrow k+1$ Assume that $u \in L^{p_k}(\mathbb{R}^d)$.

We choose r in such a way that

$$r \frac{d}{d-1} = p_k \leadsto r \frac{d}{d-1} = d + \frac{d}{d-1}k$$

$$\leadsto r = d-1+k$$

Note: at each step the constant is growing

$$\|u\|_{L^{p_k}(\mathbb{R}^d)} \leq c(d, k) \|\nabla u\|_{L^d(\mathbb{R}^d)}$$

↓
to as $k \rightarrow +\infty$
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How to prove deluxe approximation in $\mathbb{R}^d$

Step 1: regularization Consider $u * \rho_\varepsilon = u_\varepsilon$ and observe

that $u_\varepsilon \rightarrow u$ and $\nabla u_\varepsilon \rightarrow \nabla u$ in $L^p(\mathbb{R}^d)$

Step 2: we obtain the compact support

Take $v \in C^\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$



Consider $v_m(x) := v(x) \cdot \psi\left(\frac{|x|}{m}\right)$

↑
coincides with v in $B(0,1)$
and $v_m(x) \equiv 0$ outside $B(0,2m)$

Easy to see (dominated convergence) that

$$v_m \rightarrow v \quad \text{and} \quad \nabla v_m \rightarrow \nabla v \quad \text{in } L^p(\mathbb{R}^d).$$

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