

How to obtain different BC

Example 1

$$\begin{cases} \ddot{u} = u^3 \\ u(0) = 4 & \text{DBC} \\ \dot{u}(1) = 0 & \text{NBC} \end{cases}$$

$$F(u) = \int_0^1 \left\{ \frac{1}{2} \dot{u}^2 + \frac{1}{4} u^4 \right\} dx \quad L(x, s, p) = \frac{1}{2} p^2 + \frac{1}{4} s^4$$

Variational formulation

$$\min \{ F(u) : u \in H^1((0,1)) : u(0) = 4, \dot{u}(1) = 0 \}$$

↑
this makes sense
ABSOLUTELY NOT
This condition makes NO SENSE

Compactness and LSC are standard

$$M \geq F(u_m) \Rightarrow \|\dot{u}_m\|_{L^2}^2 \leq 2M$$

The DBC we have is enough to deduce $\|u_m\|_{L^\infty} \leq M'$

Regularity and NBC As usual $\varphi(t) := F(u_0 + t v)$
↑
minimizer ↑ $\in C_c^\infty$

$$\varphi'(0) = \int_0^1 (\dot{u}_0 \dot{v} + u_0^3 v) dx = 0$$

$$\leadsto \int_0^1 \dot{u}_0 \dot{v} dx = - \int_0^1 u_0^3 v dx$$

≠ 0
 $g \varphi$

$$\rightarrow (\ddot{u}_0)' = u_0^3 \quad \text{weak form of ELE}$$

From H^1 to C^2 and then C^∞ in the usual way

Now we consider $v \in C^1([0,1])$ s.t. $v(0) = 0$ and $v(1) \neq 0$.
Then

$$\begin{aligned} 0 - \varphi'(0) &= \int_0^1 \{ \ddot{u}_0 v + u_0^3 v \} dx = \\ &= \int_0^1 \{ \underbrace{-\ddot{u}_0}_{=0} + \underbrace{u_0^3}_{\neq 0} \} v dx + \underbrace{\ddot{u}_0(1)}_{\neq 0} v(1) - \underbrace{\ddot{u}_0(0)}_{=0} v(0) \end{aligned}$$

Necessarily $\ddot{u}_0(1) = 0$.

Uniqueness as in the previous lecture.
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Example 2

$$\begin{cases} \ddot{u} = u^3 \\ u(0) = 4 \\ \ddot{u}(1) = 7 \end{cases}$$

Let us recall that NBCs are of the form $\underbrace{L_p}_{p=7} = 0$

$$\rightarrow L(x, s, p) = \frac{1}{2} p^2 - 7p + \frac{1}{4} s^4$$

the weak formulation becomes

$$\min \left\{ \int_0^1 \left\{ \frac{1}{2} \dot{u}^2 - 7\dot{u} + \frac{1}{4} u^4 \right\} dx : u \in H^1((0,1)), u(0) = 4 \right\}$$

Compactness The key point is that

$$L(x, s, p) \geq \frac{1}{4} p^2 + \frac{1}{4} s^4 - A$$

use that

$$\frac{1}{2} p^2 - 7p \geq \frac{1}{4} p^2 - A \quad \forall p \in \mathbb{R}$$

$$M \geq F(u_n) \Rightarrow \|u_n\|_{L^2((0,1))}^2 \leq M' = 4M + A \cdot \text{length of interval}$$

The rest is as usual.

LSC If $u_n \rightarrow u_\infty$ then

$$\liminf_{n \rightarrow \infty} \int_0^1 \frac{1}{2} u_n^2 dx \geq \frac{1}{2} \int_0^1 u_\infty^2 dx$$

$$\lim_{n \rightarrow \infty} \int_0^1 u_n(x) dx = \int_0^1 u_\infty(x) dx$$

↑
scalar product of u_n and 1 in L^2

Genesis of NBC $\varphi(t) \dots$

$$\begin{aligned} 0 = \varphi'(0) &= \int_0^1 \{(\dot{u}_0 - 7) \dot{v} + u_0^3 v\} dx \\ &= \int_0^1 \underbrace{\{-\ddot{u}_0 + u_0^3\}}_0 v dx + [(\dot{u}_0 - 7)v]_{x=0}^{x=1} \end{aligned}$$

when we choose $v \in C^1$ with $v(1) \neq 0$ we get $\dot{u}_0(1) = 7$.

Example 3

$$\begin{cases} u'' = u^3 \\ \dot{u}(0) = 4 \\ \dot{u}(1) = 7 \end{cases}$$

Two possible approaches

→ variable change

→ modified Lagrangian

1 Variable change Choose u_0 satisfying the 2 strange NBC

$$\dot{u}_0(x) = 4 + 3x$$

$$u_0(x) = \frac{3}{2}x^2 + 4x$$

Look for a solution $u(x)$ of the form

$$u(x) = u_0(x) + v(x)$$

What is the equ. satisfied by v : $\dot{v}(0) = \dot{v}(1) = 0$

$$\ddot{v} = (u - u_0)'' = \ddot{u} - \ddot{u}_0 = u^3 - 3 = \left(\frac{3}{2}x^2 + 4x + v\right)^3 - 3$$

So we need to solve

$$\ddot{v} = \left(\frac{3}{2}x^2 + 4x + v\right)^3 - 3 \quad \text{with } \dot{v}(0) = \dot{v}(1) = 0$$

$$F(u) = \int_0^1 \left\{ \frac{1}{2} \dot{v}^2 + \frac{1}{4} \left(\frac{3}{2}x^2 + 4x + v\right)^4 - 3v \right\} dx$$

$$L(x, s, p) = \frac{1}{2} p^2 + \frac{1}{4} \left(\frac{3}{2}x^2 + 4x + s\right)^4 - 3s$$

$$\min \{ F(v) : v \in H^1((0,1)) \} \quad \text{without BCs.}$$

Compactness The key point is that

$$L(x, s, p) \geq \frac{1}{2} p^2 + \frac{1}{27} s^4 - A$$

$$\leadsto F(u_m) \leq M \Rightarrow \|u_m\|_{L^2} \leq M'$$

$$\Rightarrow \int_0^1 u_m(x)^4 dx \leq M''$$

$$\Rightarrow \exists x_m \in [0,1] \text{ s.t. } |u_m(x_m)|^4 \leq M'' \Rightarrow L^\infty \text{ estimate}$$

The rest is standard.

② Modified Lagrangian $\ddot{u}(0) = 4$
 $\ddot{u}(1) = 7$

$$L_p = 0$$

$$L_p = p - 4 - 3x$$

$$L_s = s^3$$

$$L(x, s, p) = \frac{1}{2} p^2 - (4 + 3x)p + \frac{1}{4} s^4$$

$$(L_p)' = L_s \quad [\ddot{u} - 4 - 3x]' = u^3 \leadsto \ddot{u} - 3 = u^3 \quad \text{☹}$$

The correct choice is

$$L(x, s, p) = \frac{1}{2} p^2 - (4 + 3x)p + \frac{1}{4} s^4 - 3s$$

and we minimize without boundary cond.

$$\text{Compactness} \leadsto L(x, s, p) \geq \frac{1}{27} p^2 + \frac{1}{35} s^4 - A$$

LSC $\leadsto$ if $u_n \rightarrow u_\infty$, then

$$\lim_{n \rightarrow \infty} \int_0^1 (4 + 3x) u_n(x) dx = \int_0^1 (4 + 3x) u_\infty(x) dx$$

↑
scalar prod. of
 u_n times $4 + 3x$

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