

Weak derivatives and Sobolev spacesBasic case

- dim 1
- open set $(a,b) \subseteq \mathbb{R}$
- deriv. of order 1

General case

- dim $d \geq 1$
- any open set $\Omega \subseteq \mathbb{R}^d$
- deriv. of any order

Two main approaches

- definition W → weak (good for passing to the limit)
- definition H (good for "formulae" and algebraic things)

Weak derivatives - definition W

Let u and v be in $L^1_{loc}((a,b))$. We say that v is W-weak deriv. of u if

$$\int_a^b u(x) \varphi'(x) dx = - \int_a^b v(x) \varphi(x) dx$$

$\forall \varphi \in C_c^\infty((a,b))$
 $\uparrow$ test function

Remark The weak deriv. of u plays the role of u' in the integration by parts.

Basic properties

① Uniqueness Assume v_1 and v_2 a w.d. of μ . Then

$$\int_a^b v_1(x) \varphi(x) dx = - \int_a^b \mu(x) \varphi'(x) dx = \int_a^b v_2(x) \varphi(x) dx$$

$$\Rightarrow \int_a^b (v_1(x) - v_2(x)) \varphi(x) dx = 0$$

$$\text{FLCV} \Rightarrow v_1(x) = v_2(x) \uparrow \text{almost everywhere}$$

② Linearity $(\alpha u_1 + \beta u_2)' = \alpha v_1 + \beta v_2$
if v_1 is w.d. of μ_1 and v_2 is w.d. of μ_2

③ Trivial classical compatibility If $u \in C^1((a,b))$, then $u'(x)$ is the w-weak deriv.

④ less trivial classical comp. Assume that $u \in L^1_{\text{loc}}$ admits a weak der. $v \in C^0((a,b))$
Then $u \in C^1((a,b))$

Proof. Let $U: (a,b) \rightarrow \mathbb{R}$ be such that $U'(x) = v(x)$
Then

$$- \int_a^b U(x) \varphi'(x) dx = \int_a^b v(x) \varphi(x) dx = - \int_a^b \mu(x) \varphi'(x) dx$$

$\uparrow$ classical cut. by part. $\uparrow$ v is w.d. of μ

$$\Rightarrow \int_a^b (U(x) - \mu(x)) \varphi'(x) dx = 0 \quad \forall \varphi \in C_c^\infty((a,b))$$

(DBR)

$$\Rightarrow U(x) - \mu(x) \equiv \text{constant}$$

$$\Rightarrow \mu \in C^1$$

(coincides a.e. with C^1 function)

Example 1 Assume $c \in (a, b)$, $u_1 \in C^1([a, c])$, $u_2 \in C^1([c, b])$
 Assume that $u_1(c) = u_2(c)$. Then

$$u(x) := \begin{cases} u_1(x) & x \in [a, c] \\ u_2(x) & x \in [c, b] \end{cases}$$

admits the w-weak deriv. that we expect.

$$\begin{aligned} \int_a^b u(x) \varphi'(x) dx &= \int_a^c u_1(x) \varphi'(x) dx + \int_c^b u_2(x) \varphi'(x) dx \\ &= \cancel{u_1(c)\varphi(c)} - \int_a^c u_1'(x) \varphi(x) dx \\ &\quad - \cancel{u_2(c)\varphi(c)} - \int_c^b u_2'(x) \varphi(x) dx \end{aligned}$$

Example 2 The Heaviside function

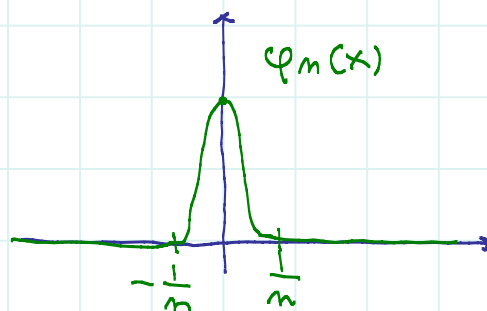
$$u(x) = \begin{cases} 0 & \text{if } x \in (-1, 0) \\ 1 & \text{if } x \in (0, 1) \end{cases}$$

does not admit a weak deriv. in $L^1_{loc}([a, b])$

Proof $\int_{-1}^1 u(x) \varphi'(x) dx = \int_0^1 \varphi'(x) dx = -\varphi(0) \quad \forall \varphi \in C_c^\infty((-1, 1))$

If v is the w.d. of u then

$$\boxed{\int_{-1}^1 v(x) \varphi(x) dx = \varphi(0)}$$



If we apply with $\varphi_m(x)$ as above we see that

$$\text{RHS} \equiv 1$$

LHS $\rightarrow 0$ by dominated convergence
 (pointwise conv. to 0 + domination by $|v(x)|$)

Stability when passing to the limit

Let $\{u_n\}$ and $\{v_n\}$ be two seq. in $L^1_{loc}(a,b)$.

Assume that

(i) v_n is the w. weak deriv. of u_n for every $n \in \mathbb{N}$

(ii) $u_n \rightarrow u_\infty$ in $L^1_{loc}(a,b)$

(iii) $v_n \rightarrow v_\infty$ in $L^1_{loc}(a,b)$ } NOT REALLY NECESSARY

Then v_∞ is the w.d. of u_∞ .

Proof. For every n and every $\varphi \in C_c^\infty(a,b)$:

$$\begin{array}{ccc} \int_a^b u_n(x) \varphi'(x) dx & = & - \int_a^b v_n(x) \varphi(x) dx \\ \downarrow & & \downarrow \\ \int_a^b u_\infty(x) \varphi'(x) dx & = & - \int_a^b v_\infty(x) \varphi(x) dx \quad \text{😊} \end{array}$$

Remark Weak convergence in $L^2(a,b)$ or even in $L^p(a,b)$ is enough for the previous result.

Theorem Assume that $u \in L^1_{loc}(a,b)$ admits a weak deriv. $v \in L^1_{loc}(a,b)$.

Then for every $x_0 \in (a,b)$ it turns out that

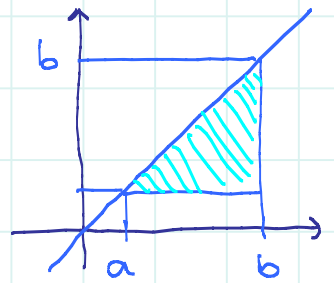
$$u(x) = u(x_0) + \int_{x_0}^x v(t) dt \quad \forall x \in (a,b)$$

Proof Let us assume for simplicity that $v \in L^1(a,b)$

$$U(x) = \int_a^x v(t) dt$$

$$\begin{aligned}
\int_a^b U(x) \varphi'(x) dx &= \int_a^b \int_a^x v(t) dt \varphi'(x) dx \\
&= \underbrace{\int_a^b \int_a^x v(t) \varphi'(x) dt dx}_{\text{over } T} \\
&= \int_a^b dt \int_t^b v(t) \varphi'(x) dx \\
&= \int_a^b v(t) dt \int_t^b \varphi'(x) dx \\
&= - \int_a^b v(t) \varphi(t) dt \\
&= - \int_a^b v(x) \varphi(x) dx \\
&\stackrel{\uparrow}{=} \int_a^b u(x) \varphi'(x) dx
\end{aligned}$$

u is
weak deriv.



$$\Rightarrow \int_a^b (U(x) - u(x)) \varphi'(x) dx = 0 \quad \text{DBR} \Rightarrow U(x) = u(x) + \text{const.}$$

If $v \in L^1(a|b)$ we have to set $U(x) = \int_{x_0}^x v(t) dt$

and distinguish two triangles

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Remark This proves that $U(x)$ "is" continuous.