

Prop. Let us consider the Dirichlet Laplacian on a reasonable domain $\Omega \subseteq \mathbb{R}^d$.

Then

→ eigenvectors are of class C^∞

→ Their gradients are orthogonal in a suitable sense.

Proof. Eigenvectors are solutions to $\Delta u = \lambda u$

Elliptic regularity: $u \in L^2 \Rightarrow u \in H^2 \Rightarrow \text{RHS} \in H^2 \Rightarrow u \in H^4$

Bootstrap $\Rightarrow u \in H^k$ for every $k \in \mathbb{N}$

$\Rightarrow u \in C^\infty$ because weak deriv. are actually true derivatives when k is large enough.

Assume now that $\Delta u = \lambda u$

$$\Delta v = \mu v$$

Claim:

$$\int_{\Omega} \langle \nabla u, \nabla v \rangle dx = 0$$

This means that $\nabla u \perp \nabla v$ as elements of $L^2(\Omega, \mathbb{R}^d)$

Indeed if we integrate by parts:

$$\int_{\Omega} \langle \nabla u, \nabla v \rangle dx = - \int_{\Omega} u \Delta v dx = - \underbrace{\mu \int_{\Omega} u v dx}_{=0 \text{ because eigenv. with diff. eigenvalues are orthogonal}}$$

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Let us consider $(0, \pi) \times (0, \pi) \subseteq \mathbb{R}^2$

Which are eigenvectors / eigenvalues of $-\Delta$ Dirichlet Laplacian

We know that $u(x, y) = \sin(nx) \sin(my)$

We observe that

$$\Delta u(x, y) = -(n^2 + m^2) u(x, y)$$

So all these functions are eigenvectors with eigenvalue $n^2 + m^2$

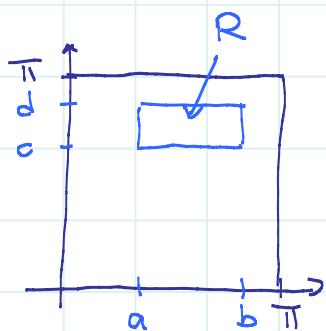
These are the only ones.

It is enough to show that they are a complete system, namely the span is dense in L^2 .

There are at least two proofs.

→ Overkill: STONE-WEIERSTRASS $\leadsto$ it is dense in cont. funct., which in turn are dense in L^2 .

→ Elementary proof: it is enough to show that we can approximate some special functions that in turn span a dense subset of L^2



It is easy to approx $\mathbb{1}_R$

$$g_n(x) \rightarrow \mathbb{1}_{(a,b)}(x)$$

$$f_m(y) \rightarrow \mathbb{1}_{(c,d)}(y)$$

$$f_n(x) \cdot g_m(y) \rightarrow \mathbb{1}_R(x, y)$$

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Now we know that

$$u \in H_0^1((0, \pi) \times (0, \pi)) \iff u = \sum_{n, m} u_{n, m} \sin(nx) \sin(my)$$

$$\text{with } \sum_{n, m} (n^2 + m^2) u_{n, m}^2 < +\infty$$

Theorem To every $u \in H_0^1((0, \pi) \times (0, \pi))$ let us associate the trace for $y = \frac{\pi}{2}$.

Then

$$u(x, \frac{\pi}{2}) \in H^{1/2}((0, \pi))$$

More important; the trace is surjective with values in $H^{1/2}$.

Proof Let us set $g(x) := u(x, \frac{\pi}{2})$.

$u \in H_0^1 \Rightarrow g \in H^{1/2}$ We observe that

$$g(x) = \sum_{n,m} u_{n,m} \sin(nx) \underbrace{\sin(m \frac{\pi}{2})}_{\substack{(-1)^k \text{ if } m=2k+1 \\ 0 \text{ if } m=2k}}$$

$$= \sum_{n,k} u_{n,2k+1} (-1)^k \sin(nx)$$

$$= \sum_n g_n \sin(nx) \quad \text{where } g_n := \sum_k (-1)^k u_{n,2k+1}$$

$$\text{Now } g \in H^{1/2} \Leftrightarrow \sum_n n g_n^2 < +\infty$$

$$\text{Observe that } g_n \leq \left(\sum_k u_{n,2k+1}^2 (n^2 + (2k+1)^2) \right)^{1/2} \left(\sum_k \frac{1}{n^2 + (2k+1)^2} \right)^{1/2}$$

so that

$$\begin{aligned} n g_n^2 &\leq \boxed{n \sum_k \frac{1}{n^2 + (2k+1)^2}} \cdot \sum_k u_{n,2k+1}^2 (n^2 + (2k+1)^2) \\ &\leq M \\ &\sim n \int_0^{+\infty} \frac{1}{n^2 + x^2} dx \end{aligned}$$

and therefore

$$\sum_n n g_n^2 \leq M \sum_n \sum_k u_{n,2k+1}^2 (n^2 + (2k+1)^2) < +\infty$$

For every $g \in H^{1/2}$, there exists $u \in H^1$ s.t. $g(x) = u(x, \frac{\pi}{2})$

We are given $g(x) = \sum_n g_n \sin(nx)$ with $\sum_n n g_n^2 < +\infty$

and we want to define $u_{n,m}$ such that

$$\sum_{n,m} u_{n,m}^2 (n^2 + m^2) < +\infty \quad \leadsto u \in H^1 \quad (*)$$

$$g_n = \sum_k (-1)^k u_{n,2k+1} \quad (**)$$

A good choice is

$$u_{n,2k+1} := (-1)^k g_n \frac{1}{n^2 + (2k+1)^2} \frac{1}{\left(\sum_k \frac{1}{n^2 + (2k+1)^2} \right)}$$

In this way (**) is almost trivial. Let us check (*)

$$\sum_{n,k} u_{n,2k+1}^2 (n^2 + (2k+1)^2)$$

$$= \sum_{n,k} g_n^2 \frac{1}{(n^2 + (2k+1)^2)^2} \frac{1}{\left(\sum_k \dots \right)^2} \cancel{(n^2 + (2k+1)^2)}$$

$$= \sum_n g_n^2 \frac{1}{\left(\sum_k \dots \right)^2} \sum_{k=1}^{\infty} \cancel{\frac{1}{n^2 + (2k+1)^2}} = \sum_n g_n^2 \frac{1}{\underbrace{\left(\sum_k \dots \right)}_{\sim \frac{1}{n}}}$$

$$= \sum_n n g_n^2 < +\infty \quad \text{--- o --- o ---}$$

WHAT WOULD REMAIN TO DO

1 Evolution problems

$$u_t = \Delta u + \sin u$$

$$u_{tt} = \Delta u + \sin u$$

2 Bounded Variation Functions (extension of $W^{1,1}(\Omega)$ where things work)

3 Dual of $C_c^\circ \rightsquigarrow$ dual is a suitable space of measures

4 Spectral theorem for unbounded operators (and also for continuous operators that are not compact)

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