

- DIRICHLET LAPLACIAN- $\ddot{u}$ $(0, \pi)$ $H^2 \cap H_0^1$ eigenvalues n^2
 $\uparrow$
 $n \geq 1$

eigenvectors

 $\sin(mx)$ (up to a mult. const.)**- NEUMANN LAPLACIAN** H^2 with NBCeigenvalues n^2
 $\uparrow$
 $n \geq 0$

eigenvectors

 $\cos(mx)$ (up to const.)
 $\uparrow$
including the case $n=0$ **- PERIODIC LAPLACIAN**Interval $(0, 2\pi)$ H^2 periodic $\uparrow u$
 $\downarrow \dot{u}$ eigenvalues n^2
 $\uparrow$
 $n \geq 0$

eigenvectors

 $\sin(mx), \cos(mx)$
all eigenspaces with $n \geq 1$
one of dim 2
($n=0 \leadsto \dim=1$)

Remark The theory is similar for derivatives of order $2k$ with k integers

(even is important in order to obtain a symmetric operator $\leadsto$ even # of int. by parts)

Remark All these operators are CLOSED, namely

$$\left. \begin{array}{l} u_n \rightarrow u_\infty \\ Au_n \rightarrow v_\infty \end{array} \right\} \Rightarrow v_\infty = Au_\infty$$

This is weaker than continuity because of the second assumption

From now on we focus on the Dirichlet case

Prop. If $A = -\text{dir. Lap.} \Rightarrow D(A^{1/2}) = H_0^1((0, \pi))$

Proof. $\Rightarrow$ Let us write $u(x) = \sum_{n=1}^{\infty} u_n \sin(nx)$

Assumption $\sum_{n=1}^{\infty} n^2 u_n^2 < +\infty$

[In abstract: $D(A^s) = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2s} u_n^2 < +\infty \right\}$
 $\lambda_n = n^2$]

$$S_m(x) : \sum_{k=1}^m u_k \sin(kx) \quad S_m(x) \rightarrow u(x) \text{ in } L^2((0, \pi))$$

$$S'_m(x) = \sum_{k=1}^m k u_k \cos(kx)$$

Now $\cos(kx)$ is (up to a constant) an orthogonal system in L^2

Therefore $\sum_{k=1}^{\infty} k u_k \cos(kx)$ converges in $L^2((0, \pi))$ if and only if $\sum_{k=1}^{\infty} k^2 u_k^2 < +\infty$ and this is the assumption.

$\leadsto S'_m(x) \rightarrow v(x)$ in $L^2((0, \pi))$ and as always $v(x) = u'(x)$

This proves that $u \in H^1((0, \pi))$. Moreover $S_m(x) \rightarrow u(x)$ uniformly in $[0, \pi]$ and since $S_m(x) = 0$ in $x=0$ and $x=\pi$ this is true also for $u(x)$, namely $u \in H_0^1$.
Here we exploited Ascoli - Arzelà.

$\Leftarrow$ Assumption: $u \in H_0^1((0, \pi))$ Aim: $\sum_{n=1}^{\infty} n^2 u_n^2 < +\infty$

$$u_n := \int_0^\pi u(x) \sin(nx) dx \quad (\text{up to constants})$$

$$= \left[-\frac{1}{n} \cos(nx) u(x) \right]_{x=0}^{x=\pi} + \frac{1}{n} \int_0^\pi \underbrace{u'(x)}_{v(x)} \cos(nx) dx$$

" 0

component of $v(x)$ wrt
the orthogonal system $\cos(nx)$
 $=: v_n$

We have proved that

$$\sum_{n=1}^{\infty} n^2 u_n^2 = \sum_{n=1}^{\infty} v_n^2 < +\infty \quad \text{because } v(x) \in L^2$$

Important point What is $A^{1/2} u$?

$$\text{If } u = \sum_{n=1}^{\infty} u_n \sin(nx) \quad \text{then} \quad A^{1/2} u = \sum_{n=1}^{\infty} \underbrace{n}_{\lambda_n^{1/2}} u_n \sin(nx)$$

$$\text{We observe that } \|A^{1/2} u\|_{L^2}^2 = \sum_{n=1}^{\infty} n^2 u_n^2$$

On the other hand

$$u' = \sum_{n=1}^{\infty} n u_n \cos(nx) \quad \text{and} \quad \|u'\|_{L^2}^2 = \sum_{n=1}^{\infty} n^2 u_n^2$$

$$\text{So it is true that } \|A^{1/2} u\|_{L^2} = \|u'\|_{L^2}$$

$$\text{but it is not true that } A^{1/2} u = u'$$

FRACTIONAL SOBOLEV SPACES

$p=2$

$$H^s((0, \pi)) = D(A^{s/2})$$

$\uparrow$
+ DBC

Prop. If $u \in H^s((0, \pi))$ with $s > \frac{1}{2}$, then u is continuous and satisfies DBC

Brutal mode: $u \in W^{m,p}$ is continuous when $\frac{mp}{s} > d$

Proof $u(x) = \sum_{n=1}^{\infty} u_n \sin(nx)$

$$u \in H^s \Leftrightarrow \sum_{n=1}^{\infty} n^{2s} u_n^2 < +\infty$$

$\lambda_n^{2\frac{s}{2}}$

Sufficient condition for u cont. is that $\sum_{n=1}^{\infty} |u_n| < +\infty$

$$\begin{aligned} \sum_{n=1}^{\infty} |u_n| &= \sum_{n=1}^{\infty} \underbrace{n^s |u_n|}_{\frac{1}{2}} \underbrace{\frac{1}{n^s}}_{\frac{1}{2}} \\ &\leq \left\{ \sum_{n=1}^{\infty} n^{2s} u_n^2 \right\}^{1/2} \underbrace{\left\{ \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right\}^{1/2}}_{\text{converges iff } 2s > 1} \end{aligned}$$

Remark We expect $u(x)$ to be holder continuous with a suitable exponent.

We expect that for $s \leq \frac{1}{2}$ u is not even in L^∞ , and therefore DBC are "lost"

Example For $s < \frac{1}{2}$ we expect that $u(x) = \frac{1}{x^a}$ is in H^s for some a .

Heuristics $\frac{1}{x^a} \in L^2((0, \pi)) \Leftrightarrow 2a < 1 \Leftrightarrow a < \frac{1}{2}$

we expect S -deriv. to be $\sim \frac{1}{x^{a+s}}$

and therefore $\frac{1}{x^a} \in H^s \Leftrightarrow 2a + 2s < 1$

$$u_m := \int_0^\pi \frac{1}{x^a} \sin(mx) dx \stackrel{\substack{\uparrow \\ mx = y \\ x = \frac{y}{m} \\ dx = \frac{1}{m} dy}}{=} \frac{1}{m} \int_0^{m\pi} \frac{m^a}{y^a} \sin(y) dy$$
$$= \frac{m^a}{m} \boxed{\int_0^{m\pi} \frac{\sin y}{y^a} dy}$$

convergent

$$u_m \sim \frac{1}{m^{1-a}}$$

$$u \in H^2 \Leftrightarrow \sum m^{2s} u_m^2 < +\infty \Leftrightarrow \sum \frac{m^{2s}}{m^{2-2a}} < +\infty$$

$$\Leftrightarrow 2 - 2a - 2s > 1$$

$$\Leftrightarrow 2a + 2s < 1$$

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