

THE EXAMPLE**DIRICHLET LAPLACIAN**

$$\ddot{u} = f$$

$$u(0) = u(\pi) = 0$$

$$(a, b) = (0, \pi)$$

$$\Delta u = f$$

$$u|_{\partial\Omega} = 0$$

$\Omega \subseteq \mathbb{R}^d$ bounded open set with
reasonable boundary

(for example a ball)

Let us consider the operator

$$I: L^2((a, b)) \rightarrow L^2((a, b))$$

$$f \rightarrow u$$

↑
unique solution

$$I: L^2(\Omega) \rightarrow L^2(\Omega)$$

↑

$$f \rightarrow u$$

Inverse of the
Laplacian

Step 1 The operator I is well-defined (existence and uniqueness)

First way Variational formulation

$$\min \left\{ \int_{\Omega} \left(\frac{1}{2} |\nabla u(x)|^2 + f(x)u \right) dx : u \in H_0^1(\Omega) \right\}$$

↑
DBC

Standard direct method applies and produces a unique solution u which is also in $H^2(\Omega)$.

Second way Lax - Milgram (because the equ. is linear)

$$-\int_{\Omega} \langle \nabla u(x), \nabla \varphi(x) \rangle dx = \int_{\Omega} f(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

Third way Only in dimension one we set

$$F(x) := \int_0^x f(t) dt$$

$$\hat{F}(x) := \int_0^x F(t) dt$$

$$u(x) = \hat{F}(x) + ax + b$$

↑ ↑
suitable values for DBC

Linearity is always trivial.

Step 2 We claim that I is symmetric

Proof In dim 1 $\ddot{u} = f$ $\ddot{v} = g$
 $I(f) = u$ $I(g) = v$

$$\begin{aligned} \langle I(f), g \rangle &= \int_0^\pi u g dx = \int_0^\pi u \ddot{v} dx = - \int_0^\pi \ddot{u} v dx = \int_0^\pi \ddot{u} v dx \\ &= \int_0^\pi f v = \langle f, I(g) \rangle \end{aligned}$$

↑ DBC on u ↑ DBC on v

In higher dimension is the same

$$\int_\Omega u g dx = \int_\Omega u \Delta v = - \int_\Omega \langle \nabla u, \nabla v \rangle = \int_\Omega \Delta u \cdot v = \int_\Omega f v$$

Step 3 We claim that I is compact.

Informally: an operator is COMPACT if "it improves the INPUT in such a way that the OUTPUT is better than the input"

Proof in dim 1 Let $\{f_n\} \subseteq L^2((0, \pi))$ be any sequence.

Consider in the corresponding solutions.

Consider $u_n = v_n$.

Then $\|v_n\|_{L^2}$ is bounded. By Rolle's theorem, there exists $x_n \in (0, \pi)$ s.t. $u_n'(x_n) = v_n'(x_n) = 0$.

In the usual way we can apply Ascoli-Arzelà to u_n and therefore

$$u_n \rightarrow u_\infty \quad \text{uniformly}$$

Now consider $\{u_n\}$. It is a sequence of Lipschitz functions (because of the L^∞ bound on v_n), and again $u_n(0) = 0$.

Again by AA

$$u_n \rightarrow u_\infty \quad \text{uniformly (and in particular in } L^2((0, \pi)) \text{)}$$

Of course $v_\infty = u_\infty$.

(All convergences are up to subsequences)

Remark $I: L^2((0, \pi)) \rightarrow H^1((0, \pi))$ Is it compact?

(Yes because we proved also unif. conv. of derivatives)

$I: L^2((0, \pi)) \rightarrow H^2((0, \pi))$ Is it compact?

(No: indeed compactness would imply that

$$\{f_n\} \subseteq L^2 \text{ bounded} \Rightarrow u_{n_k} \rightarrow u_\infty \text{ in } H^2$$

$\Downarrow$

$$u_{n_k}'' \rightarrow u_\infty'' \text{ in } L^2$$

"
 f_{n_k}

and this is not true for every f_n .

Proof in dim d $\Delta u_n = f_n$ with $\{f_n\}$ bounded in L^2

As in the regularity theory we obtain bounds on

$$\|\nabla u_n\|_{L^2} \leq M'$$

$$\|D^2 u_n\|_{L^2} \leq M''$$

$\uparrow$ all second derivatives of u_n

From Poincaré inequality we obtain also

$$\|u_n\|_{L^2} \leq M'''$$

At this point we can apply compact embeddings and deduce that

$$u_{n_k} \rightarrow u_\infty \text{ in } L^2$$

$$\nabla u_{n_k} \rightarrow \nabla u_\infty \text{ in } L^2$$

— o — o —

At the end of the day

I IS A COMPACT OPERATOR

and we can apply the spectral theorem.

By uniqueness I is also injective, and therefore

THE INVERSE OF I IS AN UNBOUNDED MULT. OPERATOR
CALLED THE DIRICHLET LAPLACIAN

$$A : D(A) \rightarrow H$$

$$\uparrow \quad \uparrow L^2(\Omega)$$

$$H^2(\Omega) \cap H_0^1(\Omega)$$

$\uparrow$ DBC

Special case in $(0, \pi)$ Let us find eigenvalues / eigenvectors

$$\ddot{u} = \lambda u$$

First of all, $\lambda < 0$:

$$\begin{aligned} u \ddot{u} &= \lambda u^2 \\ \int u \ddot{u} &= \lambda \int u^2 \\ &\quad - \int \dot{u}^2 \end{aligned}$$

$$\ddot{u} = -\alpha u \quad \text{with } \alpha > 0$$

$$u(x) = c_1 \cos(\sqrt{\alpha} x) + c_2 \sin(\sqrt{\alpha} x)$$

$$\text{we apply DBC} \rightsquigarrow c_1 = 1 \rightsquigarrow \sqrt{\alpha} = n \rightsquigarrow \alpha = n^2$$

Eigenvalues: $-n^2$ with n positive integer

Eigenvectors: $\sin(nx)$ up to a multiplicative constant if we want them to be orthonormal

Remark These are the eigenvalues of the second derivative with DBC.

The same problem with NBC produces an operator with eigenvalues $-n^2$ and eigenvectors $\cos(nx)$.

— o — o —