

## BAIRE POINT 9

## Pointwise limit of cont. functions

Theorem Let  $X$  be a COMPLETE metric <sup>space</sup> (Baire space is probably enough)

Let  $Y$  be a metric space

Let  $f_n: X \rightarrow Y$  be a sequence of cont. functions.

Let  $f_\infty: X \rightarrow Y$  be the pointwise limit, namely

$$f_n(x) \rightarrow f_\infty(x) \quad \forall x \in X.$$

Then  $f_\infty$  is continuous in a  $G_\delta$  dense set, namely the set of disc. points of  $f_\infty$  is an  $F_\sigma$ -set with empty interior.

Proof (The proof of the three nested open sets)

By a general fact the discontinuity points of  $f_\infty$  are

$$\bigcup_{\varepsilon > 0} D_\varepsilon = \bigcup_{k=1}^{\infty} D_{1/k}$$

where

$$D_\varepsilon := \{x \in X : \forall \eta > 0 \exists (y, z) \in B(x, \eta) \text{ s.t. } d_Y(f_\infty(y), f_\infty(z)) \geq \varepsilon\}$$

$\uparrow$  closed sets

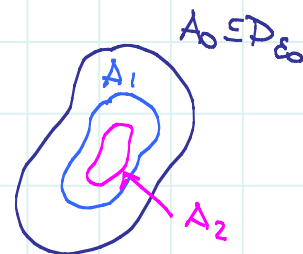
Claim:  $\text{Int}(D_\varepsilon) = \emptyset$  for every  $\varepsilon > 0$ .

Assume by contradiction that it is not the case.

Then there exists  $A_0 \subseteq D_{\varepsilon_0}$   
 $\uparrow$   
 open set  $\neq \emptyset$

We observe that  $A_0$  is a Baire space.

For every  $k \geq 1$  we consider



$$C_k := \{x \in A_0 : d_Y(f_n(x), f_m(x)) \leq \frac{1}{4} \varepsilon_0 \quad \forall n \geq k, \forall m \geq k\}$$

$$= \bigcap_{\substack{n \geq k \\ m \geq k}} \left\{ x \in A_0 : d_Y(f_n(x), f_m(x)) \leq \frac{1}{4} \varepsilon_0 \right\} \quad \nwarrow \text{closed sets}$$

[Question: why we do not define

$$C_k := \{x \in A_0 : d_Y(f_n(x), f_\infty(x)) \leq \frac{1}{4} \varepsilon \quad \forall n \geq k\}$$

We observe that  $C_k$  is a closed set for every  $k \geq 1$  and  $\bigcup_{k=1}^{\infty} C_k = A_0$  and in addition

$$d_Y(f_n(x), f_\infty(x)) \leq \frac{1}{4} \varepsilon_0 \quad \forall n \geq k \quad \forall x \in C_k$$

$\uparrow$  pass to the limit as  $n \rightarrow +\infty$

Baire category  $\Rightarrow \exists A_1 \subseteq C_{k_1}$

Now consider  $f_{k_1}$ . It is a cont. function, and therefore

$\exists A_2 \subseteq A_1$  s.t. the oscillation of  $f_{k_1}(x)$  inside  $A_2$  is  $\leq \frac{1}{4} \varepsilon_0$

Now for every  $y$  and  $z$  in  $A_2$  we have that

$$\begin{aligned} d_Y(f_n(y), f_\infty(z)) &\leq d_Y(f_\infty(y), f_{k_1}(y)) \leq \frac{1}{4} \varepsilon_0 \\ &+ d_Y(f_{k_1}(y), f_{k_1}(z)) \leq \frac{1}{4} \varepsilon_0 \\ &+ d_Y(f_{k_1}(z), f_\infty(z)) \leq \frac{1}{4} \varepsilon_0 \\ &\leq \frac{3}{4} \varepsilon_0 \end{aligned}$$

but on the other we know that LHS  $\geq \varepsilon_0$ .

Corollary Assume that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at each  $x \in \mathbb{R}$ . Then  $f'(x)$  is continuous in a  $G_\delta$  dense set.

Proof just observe that  $f'(x)$  is the pointwise limit of

$$f'_n(x) := \frac{f(x+n) - f(x)}{n} \quad \uparrow \text{continuous function}$$

$\uparrow n = 1/m$

# UNBOUNDED OPERATORS (in HILBERT SPACES)

Def Let  $H$  be a Hilbert space. An unbounded operator is a linear function

$$A : D(A) \rightarrow H$$

↑  
vector subspace of  $H$ , called  
the DOMAIN of  $A$

Achtung! The term is misleading because UNBOUNDED does not imply NOT BOUNDED.

For example, the operator  $Ax \equiv 0$  defined on a subspace of  $H$ , or even with  $D(A) = H$ , is an unbounded operator 😊

In other words:

UNBOUNDED = we do not ask a priori a bound.

Def. (Multiplication operator)

Let us assume that  $H$  is separable. We say that  $A$  is a MULTIPLICATION OPERATOR if there exists

→ a Hilbert basis  $\{e_n\}$

→ a sequence of real numbers  $\{\lambda_n\}$

such that

$$A e_n = \lambda_n e_n$$

In words, these operators are the infinite dim. versions of diagonal matrices

The operator is extended by linearity, so if

$$u = \sum_{n=1}^{\infty} u_n e_n$$

then  $Au = \sum_{n=1}^{\infty} \lambda_n u_n e_n$

The series converges if and only if  $\sum_{n=1}^{\infty} \lambda_n^2 u_n^2 < +\infty$  and therefore

$$D(A) := \left\{ u \in H : \sum_{n=1}^{\infty} \lambda_n^2 u_n^2 < +\infty \right\}$$

If  $\{\lambda_n\}$  is bounded, then  $D(A) = H$

If  $\{\lambda_n\}$  is unbounded, then  $D(A)$  is strictly contained in  $H$ .

**STANDARD EXAMPLE** Let  $A: H \rightarrow H$  be as in the spectral thm.

→ linear

→ continuous and symmetric

→ compact.

Assume that  $A$  is injective, namely 0 is not an eigenvalue.

Then both  $A$  and  $A^{-1}$  are multiplication operators.

Because of the spectral theorem

$$A e_n = \lambda_n e_n$$

↑ bounded and  $\lambda_n \rightarrow 0$

$$A^{-1} e_n = \frac{1}{\lambda_n} e_n$$

↑ unbounded sequence

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## POWERS OF MULTIPLICATION OPERATORS

Given a mult. op.  $A: D(A) \rightarrow H$ , we can consider the power

$$A^s: D(A^s) \rightarrow H$$

defined by

$$A^s e_n = \lambda_n^s e_n$$

with domain

$$D(A^s) := \left\{ u \in H : \sum_{n=1}^{\infty} \lambda_n^{2s} u_n^2 < +\infty \right\}$$

What about  $s$ ?

→ If  $s \in \mathbb{N} \setminus \{0\}$ , then no restriction

→ If  $s \in \mathbb{Z}$ , it is enough that  $\lambda_n \neq 0$

→ If  $s \in \mathbb{R}$ , we have to restrict to  $\lambda_n > 0$   
(positive operators)

**Remark** We have the classical composition rule, namely

$$A^s (A^r u) = A^{s+r} u \quad \forall u \in \text{suitable domain}$$

Def If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is any function, we can define  $f(A)$  as the unbounded operator such that

$$f(A) e_n = f(\lambda_n) e_n \quad \forall n \geq 1$$

With this rule PRODUCT OF FUNCTIONS becomes COMPOSITION OF OPERATORS, namely

$$g(f(A)) = (g \circ f)(A)$$

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