

BAIRE POINT 8

Open mapping theorem and related issues

General problem Given $f: V \rightarrow W$, and given $w \in W$, solve the equation

$$f(v) = w$$

Defn Setting as above.

(1) A QUALITATIVE SOLVER is a function $S: W \rightarrow V$ such that

[Added after video]

$$f(S(w)) = w \quad \forall w \in W$$

$S(w)$ is a possible solution of $f(v) = w$

(2) A QUANTITATIVE SOLVER is a solver that satisfies

$$\exists M \in \mathbb{R} \quad \text{s.t.} \quad \|S(w)\|_V \leq M \|w\|_W \quad \forall w \in W$$

(3) A quantitative solver on a dense subset is a function

$\hat{S}: \hat{D} \rightarrow V$, where $\hat{D}$ is dense in W and

$$f(\hat{S}(w)) = w \quad \forall w \in \hat{D}$$

$$\exists \hat{M} \in \mathbb{R} \quad \text{s.t.} \quad \|\hat{S}(w)\|_V \leq \hat{M} \|w\|_W \quad \forall w \in \hat{D}.$$

Questions Existence of

Q1 a solver

Q2 a linear solver

Q3 a quantitative solver

Q4 a linear quantitative solver

Answer to Q1 Let $f : V \rightarrow W$ be a function
 $\uparrow \quad \uparrow$
set set

Then a solver exists if and only if f is surjective

Proof Well-known set theoretic fact (S is a one-sided inverse of f).

Answer to Q2 Let $f : V \rightarrow W$ be a linear map.
 $\uparrow \quad \uparrow$
vector spaces

Then a linear solver exists if and only if f is surjective.

Proof

- Choose an algebraic basis $\{w_i\}_{i \in I}$ of W
- For every $i \in I$, choose $v_i \in V$ s.t. $f(v_i) = w_i$
- Define $S(w_i) = v_i$
- Extend S to W by linearity
- Check that it works.

Answer to Q3 Is contained in three statements.

Prop 1 (Characterization of quantitative solvers)

Let $f : V \rightarrow W$ be a linear map (we do NOT need continuity)
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Then the following are equivalent.

- (1) Existence of a quantitative solver (not nec. linear)
- (2) f is OPEN (namely $f(\text{open subset of } V) = \text{open in } W$)
- (3) $\exists R > 0$ s.t. $f(B_V(0, R)) \supseteq B_W(0, 1)$. (exchange stat.)

Remark. What we actually prove is that

$$(1) \Leftrightarrow (3)$$

$$(2) \Leftrightarrow (3)$$

Prop 2 (Weaker characterization)

Assume $f: V \rightarrow W$ linear and continuous
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Then the following are equivalent.

- (1) Existence of a quantitative solver
- (2) " " " " " on a dense subset
- (3) $\exists R > 0$ s.t. $\text{Clos}_W (f(B_V(0, R))) \supseteq B_W(0, 1)$
 $\uparrow \uparrow \uparrow$

Remark What we actually prove is that

$$(1) \Leftrightarrow (2)$$

(2) is the exchange statement)

$$(2) \Leftrightarrow (3)$$

Theorem (Open mapping thm)

Assume that $f: V \rightarrow W$ is linear and continuous
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Then the following facts are equivalent.

- (1) f IS OPEN (which in turn is equivalent to the existence of a quantitative solver)
- (2) f IS surjective.

The open mapping has three corollaries.

COROLLARY 1 (Continuity of the inverse function)

Assume that $f: V \rightarrow W$ is linear, continuous and invertible
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Then the inverse function is linear and continuous.

Proof Linearity is a general fact.

The continuity of the inverse is equivalent to f being open.

COROLLARY 2 (Equivalence of norms)

Let V be a vector space, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in V .

Assume that

(i) V is a Banach space wrt **both norms**

(ii) there exists $M \in \mathbb{R}$ s.t.

$$\|v\|_2 \leq M \|v\|_1 \quad \forall v \in V$$

Then

$$\exists \hat{M} \text{ s.t. } \|v\|_1 \leq \hat{M} \|v\|_2 \quad \forall v \in V$$

(and therefore the two norms are equivalent)

Proof. Consider $\text{Id} : V_1 \rightarrow V_2$. This map is linear, cont. and invertible because of (ii)

Therefore, also the inverse is continuous, and hence Lip. cont., ...

COROLLARY 3 (Closed graph theorem)

Assume that $f : V \rightarrow W$ is linear

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Then f is continuous if and only if the graph of f is a closed subset of $V \times W$.

Proof. The nontrivial implication is closed graph $\Rightarrow$ cont.

Consider two norms on V :

$$\|v\|_1 := \|v\|_V$$

$$\|v\|_2 := \|v\|_V + \|f(v)\|_W$$

It is clear that $\|v\|_1 \leq \|v\|_2$ for every $v \in V$.

Claim: V is a Banach space wrt norm 2.

Assume $\{v_n\}$ is a Cauchy seq. wrt norm 2. This implies that $\{v_n\}$ is a Cauchy seq. wrt norm 1 and $\{f(v_n)\}$ is a Cauchy seq. in W .

Therefore $u_m \rightarrow u_\infty \in V$

$$f(u_m) \rightarrow w_\infty \in W$$

But $(u_\infty, w_\infty) \in \text{graph of } f$ (because graph is closed)
and therefore $w_\infty = f(u_\infty)$.

At this point $u_m \rightarrow u_\infty$ also wrt norm 2.

Now the two norms are equivalent, namely

$$\|u\|_V + \|f(u)\|_W \leq \hat{M} \|u\|_V$$

which implies that f is continuous.

Remark If $f: X \rightarrow Y$ it is **NOT TRUE** that
 $\uparrow \quad \uparrow$
metric spaces

f is cont. $\Leftrightarrow$ graph is closed.

Proof of Prop 1

(1) $\Rightarrow$ (3) Claim $f(B_V(0, M)) \supseteq B_W(0, 1)$
 $\uparrow$
constant of quantitative solver

If $w \in B_W(0, 1)$, then take $u := S(w)$ so that $f(u) = w$
and $\|u\|_V = \|S(w)\|_V \leq M \|w\|_W < M$

(3) $\Rightarrow$ (1). Take $w \in W$. Consider $\frac{w}{2\|w\|_W} \in B_W(0, 1)$

$$\Rightarrow \exists u \in B_V(0, R) \text{ s.t. } f(u) = \frac{w}{2\|w\|_W}$$

Now define $S(w) := 2\|w\|_W \cdot u$ and observe that

$$\|S(w)\| \leq \underbrace{2\|u\|_W}_{2R} \cdot \|w\|_W$$

(2) $\Rightarrow$ (3) If f is open, then $f(B_V(0,1)) \supseteq B_W(0,r_0)$
for some $r_0 > 0$

By linearity

$$f(B_V(0, \frac{1}{r_0})) \supseteq B_W(0, 1)$$

$\uparrow R$

$\uparrow$ [Connected after video]

(3) $\Rightarrow$ (2) If $f(B_V(0,R)) \supseteq B_W(0,1)$ then

$$f(B_V(x_0, r_0)) \supseteq B_W(f(x_0), \frac{r_0}{R})$$

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$$f(x_0 + r_0 B_V(0,1)) = f(x_0) + \frac{r_0}{R} f(B_V(0,R))$$

$$\supseteq f(x_0) + \frac{r_0}{R} B_W(0,1)$$

$$= B_W(f(x_0), \frac{r_0}{R}).$$

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