

Proof of BS, equivalence version (2) $\Rightarrow$ (1) is trivial, so we prove the converse

Hyp: $\forall v \in V \quad \sup_{i \in I} \|L_i(v)\|_W < +\infty$

Th: $\forall v \in V \quad \|L_i(v)\|_W \leq M \|v\|_V$
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$$C_k := \{v \in V: \forall i \in I \quad \|L_i(v)\| \leq k\}$$

$$= \bigcap_{i \in I} \underbrace{\{v \in V: \|L_i(v)\| \leq k\}}_{\text{closed set because } L_i \text{ is cont.}}$$

By assumption $\bigcup_{k=1}^{\infty} C_k = V$. Baire lemma $\Rightarrow \exists \bar{B}_r(v_0, r_0) \subseteq C_{k_0}$

Claim: $\bar{B}_r(0, 1) \subseteq C_{\frac{2k_0}{r_0}}$

Consider $v \in \bar{B}_r(0, 1)$ and write it as

$$v = \frac{1}{r_0} \underbrace{(v_0 + r_0 v)}_{\in \bar{B}_r(v_0, r_0)} - \frac{1}{r_0} \underbrace{v_0}_{\in \bar{B}_r(v_0, r_0)}$$

and therefore

$$\|L_i v\|_W \leq \frac{1}{r_0} \|L_i(v_0 + r_0 v)\|_W + \frac{1}{r_0} \|L_i v_0\|_W \leq \frac{2k_0}{r_0} \quad M$$

$\leq k_0 \qquad \leq k_0$

We conclude because $\|L_i\|_{\infty(V, W)}$ is the sup of LHS for $v \in \bar{B}_r(0, 1)$.

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Proof of BS, alternative version

(2) is $\exists A \subseteq V \quad \forall v \in A \quad \sup_{i \in I} \|L_i(v)\|_W = +\infty$
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 G_δ dense subset

We observe that

$$\{v \in V : \sup_{i \in I} \|L_i(v)\|_W = +\infty\} = \bigcap_{k=1}^{\infty} \underbrace{\{v \in V : \sup_{i \in I} \|L_i(v)\|_W > k\}}_{\text{open set } A_k}$$

This implies that the LHS is always a G_δ set. If A_k is dense for every k , then the LHS is dense.

If (2) is false, then A_{k_0} is NOT dense for some $k_0 \geq 1$, and therefore $\underbrace{V \setminus A_{k_0}}_{\substack{\text{C}_{k_0} \text{ in the previous proof} \\ \text{---} \circ \text{---} \circ \text{---}}}$ has nonempty interior

BAIRE POINT 7-bis

Failure of pointwise conv. of Fourier series of continuous functions.

Light version

Let V be the space of 2π -periodic continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ with the sup norm.

Then there exists $f \in V$ s.t. the Fourier series of f does not converge in $x=0$.

Strong version

There exists a G_δ dense subset $\mathcal{F} \subseteq V$ such that for every $f \in \mathcal{F}$ the F. series of f does not converge in a G_δ dense set of points (depending on f)

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in the sense that partial sums are NOT bounded

Proof of light version Let us define

$$S_n^f(x) := \sum_{k=0}^n \alpha_k \cos(kx) + \sum_{k=1}^n \beta_k \sin(kx)$$

↑ ↑
formulae for α_k and β_k

Consider the sequence of operators $L_m: V \rightarrow \mathbb{R}$ defined as

$$L_3(f) := \sum_{k=0}^n \alpha_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx + \frac{1}{\pi} \sum_{k=1}^n \int_0^{2\pi} f(x) \cos(kx) dx$$

If the series converges in $x=0$ for every $f \in V$, then L_n is qualitatively bounded.

This implies a quantitative bound, namely

$$|L_n(f)| \leq M \|f\|_{L^\infty}$$

On the other hand, L_m can be represented as

$$L_m(f) = \int_0^{2\pi} D_m(t) f(t) dt$$

↑ Dirichlet kernel

where $D_n(t) := \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})}$

Now we know that

$$|L_m(\varphi)| \leq \|\varphi\|_{L^8} \|D_m\|_{L^1}$$

Claim: we have equality and $\|D_n\|_{L^1} \rightarrow \infty$ as $n \rightarrow \infty$.

In order to show $\Rightarrow$ we would like to choose $f(t) = \text{sign}(D_m(t))$ but we need $f(t)$ to be continuous. To fix the problem, we approximate $\text{sign}(D_m(t))$ with continuous functions and we

pass to the limit by dominated convergence.

GENERAL FACT If $g \in L^1((a,b))$, then the norm of the operator

$$L(f) := \int_a^b f(t) g(t) dt$$

is $\|g\|_{L^1((a,b))}$ both if the domain is $L^\infty((a,b))$, and if the domain is $C^0([a,b])$.

The computation of $\|D_n\|_{L^1}$ is an exercise

$$\begin{aligned} \int_0^{2\pi} \frac{|\sin(n+\frac{1}{2})t|}{\sin \frac{t}{2}} dt &\geq 2 \int_0^{2\pi} \frac{|\sin(n+\frac{1}{2})t|}{t} dt \\ &\quad \uparrow \sin \frac{t}{2} \leq \frac{t}{2} \\ &= 2 \int_0^{2n\pi+\pi} \frac{|\sin(y)|}{y} dy \\ &\quad \uparrow (n+\frac{1}{2})t = y \\ &\quad (n+\frac{1}{2})dt = dy \\ &\quad \downarrow 2 \int_0^{+\infty} \frac{|\sin(y)|}{y} dy = +\infty \end{aligned}$$

Proof of strong version

From BS in alternative version we know that there exists a dense G_δ set in V such that

$$|S_n^f(0)|$$

is unbounded for every f in this set.

In the same way we obtain a dense G_δ set $\mathcal{F}$ in V such that

$$|S_n^f(x)| \text{ is unbounded for every } x \in \mathbb{Q}$$

(Here we use that countable int. of dense G_δ sets is again a dense G_δ set)

So for every $f \in \mathcal{F}$ and $x \in \mathbb{Q}$ $|S_n^f(x)|$ is unbounded.

Is it unbounded for more values of x ? YES! It is unbounded on a G_δ set of points x (which is dense because it contains $\mathbb{Q}$)

GENERAL FACT Let $\varphi_n: (a,b) \rightarrow \mathbb{R}$ be cont. functions.

Then

$$\{x \in (a,b) : \sup_{n \geq 1} |\varphi_n(x)| = +\infty\}$$

is a G_δ set

Indeed it can be written as

$$\bigcap_{k=1}^{\infty} \{x \in (a,b) : \sup_{n \geq 1} |\varphi_n(x)| > k\}$$

open set

Alternative proof: the sup of cont. functions is LSC and the set where it is $+\infty$ is the intersection of the sets where it is $> k$, and these are open sets because the complement where it is $\leq k$ is closed.

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