

BAIRE POINT 5 There do NOT exist a Banach space with COUNTABLE ALGEBRAIC basis.

Proof Let us assume that $u_1, \dots, u_m, \dots$ is a countable basis.
Define

$$C_k := \text{Span}(u_1, \dots, u_k) \leftarrow \text{Finite dim. vector subspace} \\ (\Rightarrow \text{closed})$$

Again $V = \bigcup_{k=1}^{\infty} C_k$
 $\uparrow$
 whole space

Baire category $\Rightarrow$ there exists C_{k_0} with $\text{Int}(C_{k_0}) \neq \emptyset$,
namely $B_V(x_0, r_0) \subseteq C_{k_0}$.

This is not possible because

$$\begin{array}{c} x_0 + \varepsilon u_{k_0+1} \\ \uparrow \\ \in C_{k_0} \end{array}$$

For ε small would be in C_{k_0} , which is impossible.
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Lemma Let V be a normed space, and let $W \subseteq V$ be a subspace with finite dimension.

Then W is closed.

Proof #1 Let $w_1, \dots, w_m$ be a basis of W .

On W we define two norms

$\rightarrow$ norm #1 : restriction of the norm of V

$\rightarrow$ norm #2 : write $w \in W$ as $c_1 w_1 + \dots + c_m w_m$ and define

$$\|w\| = (c_1^2 + \dots + c_m^2)^{1/2}$$

General fact : any two norms on a finite dim. vector space are equivalent. This is enough to conclude because

if $u_n \rightarrow u_\infty$ with $u_n \in W$, then $\{u_n\}$ is a Cauchy seq.

wt to the first norm, therefore it is a Cauchy seq. with respect to the second norm, but W is complete wrt norm #2, and therefore it is complete wrt norm #1.

Proof #2 Assume that $v_k \rightarrow v_\infty$ is a sequence with $v_k \in W$ for every $k \geq 1$. Claim: $v_\infty \in W$.
For every $k \geq 1$ write

$$v_k = c_1^k w_1 + \dots + c_m^k w_m$$

We hope that

$$c_i^k \rightarrow c_i^\infty \quad \text{as } k \rightarrow \infty,$$

at least up to subsequences. This is trivial if $\{c_i^k\}_{k \geq 1}$ are bounded for every $i = 1, \dots, m$. Assume it is not the case.

wlog assume that $|c_1^k| \geq |c_i^k|$ for every k and i and $|c_1^k| \rightarrow \infty$.
Then

$$\frac{1}{c_1^k} v_k = w_1 + \frac{c_2^k}{c_1^k} w_2 + \dots + \frac{c_m^k}{c_1^k} w_m$$

$\downarrow$ (v_k is bounded and $|c_1^k| \rightarrow \infty$)
0

$\downarrow$ α_2

$\downarrow$ α_m

(up to subsequences because they are bounded)

and in this way we have a nontrivial lin. comb. of $w_1, \dots, w_m$ which is 0.

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BAIRE POINT 6

Theorem There exist a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ that is not differentiable at every $x \in \mathbb{R}$.

Proof Consider $V =$ space of continuous and bounded functions with the uniform norm (L^∞).
This is a complete metric space.

Claim: the set of elements of V that are diff. in at least one point are a countable union of closed sets with empty interior.

This implies that there are "many" counterexamples.

$$\exists x_0 \in \mathbb{R} \text{ s.t. } \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) \in \mathbb{R}$$

QUALITATIVE!

quantitative position
↓

quantitative limit
↑

$$C_k := \{ f \in V : \exists x_0 \in [-k, k] \quad \forall h \in [-\frac{1}{k}, \frac{1}{k}]$$

$$|f(x_0+h) - f(x_0)| \leq k |h| \}$$

↑
quantitative bound on the derivative

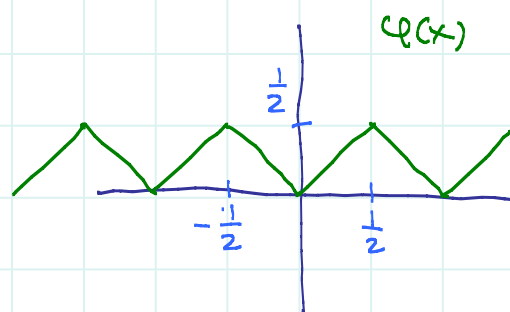
The set of NON counterexamples is $\bigcup_{k=1}^{\infty} C_k$.

Claim: $\text{Int}(C_k) = \emptyset$.

Assume it is NOT empty, namely $\exists B_r(f_0, r_0) \subseteq C_{k_0}$

Step 1 We can assume, up to reducing r_0 , that f_0 is rather smooth, for example Lip. cont., or even of class $C^{2,2}$.
(just regularize f_0)

Step 2 Consider $\varphi(x)$ as in the figure and observe that, if we set



$$\varphi_m(x) := \frac{1}{m} \varphi(m^2 x)$$

then for every $x \in \mathbb{R}$ it turns out that

$$\lim_{h \rightarrow 0^+} \frac{\varphi_m(x+h) - \varphi_m(x)}{h} = \pm m \quad (\text{depending on } x)$$

Define

$$f_n(x) = \underset{\substack{\uparrow \\ \text{center of} \\ \text{the ball}}}{f_0(x)} + \underset{\substack{\uparrow \\ \text{small in } V \text{ if } n \\ \text{is large enough}}}{\varphi_n(x)}$$

$\Rightarrow f_n \in C_k$ if n is large enough.

On the other hand

$$\boxed{\frac{\varphi_n(x_0 + a) - \varphi_n(x_0)}{a}} = \boxed{\frac{f_n(x_0 + a) - f_n(x_0)}{a}} - \boxed{\frac{f_0(x_0 + a) - f_0(x_0)}{a}}$$

tends to $\pm \infty$ Bounded by k_0 if $a \in [-\frac{1}{k_0}, \frac{1}{k_0}]$ Bounded by the lip const of f_0

We had to check that C_k are closed sets wrt uniform conv.

Assume that $f_n \rightarrow f_0$ unif.

Assume that

$$|f_n(\underset{\substack{\uparrow \\ \text{depends on } n}}{x_n + a}) - f_n(\underset{\uparrow}{x_n})| \leq K|a| \quad \forall a \in [-\frac{1}{K}, \frac{1}{K}]$$

Since $x_n \in [-k, k]$, up to subsequences $x_n \rightarrow x_\infty \in [-k, k]$ and due to unif. conv. we can pass to the limit (double limit) and obtain

$$\underbrace{|f_\infty(x_\infty + a) - f_\infty(x_\infty)|}_{= 0} \leq K|a| \quad \forall a \in [-\frac{1}{K}, \frac{1}{K}]$$

BAIRE POINT 7

BANACH - STEINHAUS THEOREM

Setting: V is a Banach space, W is a normed space, I is an index set.

For every $i \in I$, we consider a linear map

$$L_i: V \rightarrow W$$

that is also continuous

Statement 1 (Equivalence)

The following two facts are equivalent

(1) (Qualitative boundedness)

$$\forall v \in V \quad \sup_{i \in I} \|L_i(v)\|_W < +\infty$$

(2) (Quantitative boundedness)

$$\exists M \in \mathbb{R} \text{ s.t. } \|L_i(v)\|_W \leq M \|v\|_V$$

(this is equivalent to $\sup_{i \in I} \|L_i\|_{\infty(V,W)} < +\infty$)

Statement 2 (Alternative)

The following two facts are alternative.

(2) (Quantitative boundedness)

(1) There exists a G_δ dense subset $A \subseteq V$ such that

$$\sup_{i \in I} \|L_i(v)\|_W = +\infty \quad \forall v \in A$$

Corollary Let $L_n : V \rightarrow W$ be a sequence of lin. cont. operators.

Assume that $L_n(v) \rightarrow L_\infty(v)$ for every $v \in V$.

Then L_∞ is linear and continuous.

Proof Pointwise conv. $\Rightarrow$ qualitative boundedness $\Rightarrow$ quantitative boundedness $\Rightarrow$ equi-lip. continuity.

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