

BAIRE POINT 2

Does there exist a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is discontinuous in x if and only if $x \in \mathbb{R} \setminus \mathbb{Q}$?

Remark There DOES exist a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is discontinuous exactly in $\mathbb{Q}$.

Defn Let X be a top. space. A subset $Y \subseteq X$ is called

- an F_σ set if it is the countable union of closed subsets
 $\uparrow$ fermée $\uparrow$ sum = union
- a G_δ set if it is the countable intersection of open sets
 $\uparrow$ GEBIET

Lemma 1 Let X be a metric space, and let $f: X \rightarrow Y$ be a function
 $\uparrow$ another metric space

Then the set of discontinuity points of f is a F_σ -set.

Proof Let us define

$$D_\varepsilon := \{x \in X : \forall \eta > 0 \exists (y, z) \in B(x, \eta) \text{ s.t. } d_Y(f(y), f(z)) \geq \varepsilon\}$$

It is possible to check that $\bigcup_{\varepsilon > 0} D_\varepsilon$ is exactly the set of discontinuity points of f
 $\uparrow$ " ∞ "
 $\bigcup_{n=1}^{\infty} D_{1/n}$

More important, D_ε is a closed set for every $\varepsilon > 0$. Indeed, if $x_n \rightarrow x_\infty$ in X , then consider any ball $B(x_\infty, \eta)$

For n large enough

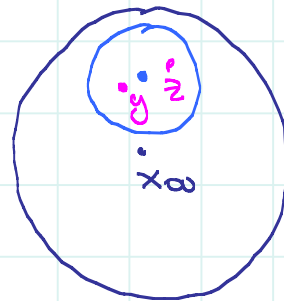
$$B(x_n, \frac{r}{2}) \subseteq B(x_\infty, 1)$$

↑

contains two points y and z with

$$d_Y(f(y), f(z)) \geq \varepsilon.$$

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Lemma 2 $\mathbb{R} \setminus \mathbb{Q}$ is not a F_σ subset of $\mathbb{R}$.

Proof Assume that

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} C_n$$

↑
closed subsets

Then $\text{Int}(C_n) \subseteq \text{Int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$ for every $n \geq 1$.

Now observe that

$$\mathbb{R} = \underbrace{\bigcup_{n=1}^{\infty} C_n \cup \bigcup_{q \in \mathbb{Q}} \{q\}}_{\text{we have written } \mathbb{R} \text{ as a countable union of closed subsets with empty interior.}}$$

we have written $\mathbb{R}$ as a countable union of closed subsets with empty interior.

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BAIRE POINT 3 Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. Let us assume that

(i) f is separately continuous in the two variables, namely

$$\forall y \in \mathbb{R} \quad x \rightarrow f(x, y) \text{ is cont. wrt } x$$

$$\forall x \in \mathbb{R} \quad y \rightarrow f(x, y) \text{ is cont. wrt } y$$

(ii) there exists $D \subseteq \mathbb{R}^2$ dense s.t.

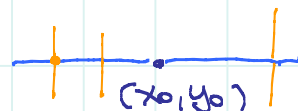
$$f(x, y) = 0 \quad \forall (x, y) \in D$$

Then $f(x, y) = 0$ for every $(x, y) \in \mathbb{R}^2$.

Remark This is simple if $D = \mathbb{Q}^2$, but in general it might happen that the intersection of D with every line $\parallel$ to the axes contains at most point.

Proof Let us assume it is not true. wlog $f(x_0, y_0) > 0$

$\exists \delta > 0$ s.t. $f(x, y_0) > 0$ for every $x \in (x_0 - \delta, x_0 + \delta)$



For every such x , there exists $r(x)$ such that $f(x, y) > 0 \quad \forall y \in [y_0 - r(x), y_0 + r(x)]$

We need $r(x)$ to be in some sense UNIFORM wrt x .

Let us make it QUANTITATIVE.

Assume that $f(x_0, y_0) \geq 4a$.

Assume that

$$f(x, y_0) \geq 2a \quad \forall x \in [x_0 - \delta, x_0 + \delta]$$

Define

$$C_k := \{x \in [x_0 - \delta, x_0 + \delta] : f(x, y) \geq a \quad \underbrace{\forall y \in [y_0 - \frac{1}{k}, y_0 + \frac{1}{k}]}_{\text{QUANTITATIVE}}\}$$

The union of C_k 's is $[x_0 - \delta, x_0 + \delta]$. They are closed. Indeed assume that

$$C_k \ni x_n \rightarrow x_\infty$$

Then for every $y \in [y_0 - \frac{1}{k}, y_0 + \frac{1}{k}]$ it turns out that

$$f(x_\infty, y) = \lim_{\substack{\uparrow \\ \text{cont. wrt } x}}_{n \rightarrow \infty} \underbrace{f(x_n, y)}_{\geq a} \geq a$$

Baire Lemma $\Rightarrow \exists k_0 \geq 1$ s.t. $C_{k_0} \ni$ interval (c, d) and $(c, d) \times (y_0 - \frac{1}{k_0}, y_0 + \frac{1}{k_0})$ intersects D .

BAIRE POINT 4

Theorem (Hilbert version) Let H be a Hilbert space.

Let us assume that $u_m \rightarrow u_\infty$ weakly in H .

Then $\{u_m\}$ is bounded

Proof For every $v \in H$ we know that $\langle u_m, v \rangle \rightarrow \langle u_\infty, v \rangle$ and in particular

$|\langle u_m, v \rangle|$ is bounded

QUALITATIVE

Define

$$C_k := \{v \in H : |\langle u_m, v \rangle| \leq k \quad \forall m \in \mathbb{N}\}$$

$$= \bigcap_{m=1}^{\infty} \underbrace{\{v \in H : |\langle u_m, v \rangle| \leq k\}}_{\text{closed set}}$$

Again H is the union of the closed sets C_k , and therefore there exists

$$\overline{B}(w_0, r_0) \subseteq C_{k_0}$$

Claim:

$$|\langle u_m, w \rangle| \leq \frac{2k_0}{r_0} \quad \forall m \geq 1 \quad \forall w \in \overline{B}(0, 1)$$

Indeed

$$w = \frac{1}{r_0} \underbrace{(r_0 w + w_0)}_{\in \overline{B}(w_0, r_0)} - \frac{1}{r_0} w_0$$

and therefore

$$|\langle u_m, w \rangle| \leq \frac{1}{r_0} \underbrace{|\langle u_m, r_0 w + w_0 \rangle|}_{\leq k_0} + \frac{1}{r_0} \underbrace{|\langle u_m, w_0 \rangle|}_{\leq k_0}$$

which proves the claim.

Now it is enough to set $w := \frac{u_m}{\|u_m\|} \in \overline{B}(0,1)$ and obtain

$$\|u_m\| \leq \frac{2k_0}{r_0} \quad \forall m \geq 1$$

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Remark The same idea works in normed spaces, namely
 $u_m \rightarrow u_\infty$ weakly in $V \Rightarrow \{u_m\}$ is bounded

Proof ...

$$C_k := \{f \in V' : |f(u_m)| \leq k\}$$

... $V' = \bigcup_{k=1}^{\infty} C_k$ and C_k is closed ...

$$\exists \overline{B}_{r_1}(f_0, r_0) \subseteq C_{k_0}$$

Claim: $|f(u_m)| \leq \frac{2k_0}{r_0} \quad \forall m \geq 1 \quad \forall f \in \overline{B}_{r_1}(0,1)$

...
$$f = \frac{1}{r_0} \underbrace{(r_0 f + f_0)}_{\overline{B}_{r_1}(f_0, r_0)} - \frac{1}{r_0} \underbrace{f_0}_{\text{...}}$$

...

Conclude by taking $f =$ aligned functional of u_m .
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