

BAIRE SPACES

Prop Let X be a topological space.

The following properties are equivalent

(i) If $\{A_i\}_{i \geq 1}$ is a sequence of OPEN DENSE subsets, then

$$\bigcap_{i=1}^{\infty} A_i \text{ is DENSE}$$

(ii) If $\{C_i\}_{i \geq 1}$ is a sequence of CLOSED subsets with $\text{Int}(C_i) = \emptyset$ then

$$\bigcup_{i=1}^{\infty} C_i \text{ has EMPTY interior}$$

(iii) If $\{A_i\}_{i \geq 1}$ is a seq. of closed subsets such that

$$\text{Int} \left(\bigcup_{i=1}^{\infty} C_i \right) \neq \emptyset$$

then there exists $i_0 \geq 1$ s.t. $\text{Int}(C_{i_0}) \neq \emptyset$

Proof (i) $\Rightarrow$ (ii) consider the complement set

$$(ii) \Leftrightarrow (iii) \text{ contrapositive } (A \Rightarrow B \Leftrightarrow \text{NOT } B \Rightarrow \text{NOT } A)$$

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Defn A Baire space is a top. space that satisfies any of the equivalent statements.

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Further notations A subset $Y \subseteq X$ is called

- **NOWHERE DENSE** if $\text{Int}(\text{Clos}(Y)) = \emptyset$ (very very small from the topological point of view)
- **MEAGER** or **FIRST CATEGORY** if it is the countable union of nowhere dense subsets (small from the top. point of view)
- **RESIDUAL** if $X \setminus Y$ is meager.

Prop In a Baire space a meager set has empty interior

Prop In any space the countable union of meager sets is again meager

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THREE MAIN CLASSES OF BAIRE SPACES

- ① COMPLETE METRIC SPACES
- ② LOCALLY COMPACT TOP. SPACES (maybe HAUSDORF)
- ③ OPEN SUBSETS OF BAIRE SPACE

Theorem (Baire Category Theorem) If X is a complete metric space, then X is a Baire space.

Proof we use the characterization (i). So let $\{A_i\}_{i \geq 1}$ be a sequence of dense open sets.

We prove that the intersection is dense, namely every ball $B(x, r)$ intersects the intersection.

Idea: we construct a sequence of balls $B(x_n, r_n)$ such that $B(x_0, r_0) = B(x, r)$ and then

$$\overline{B(x_{n+1}, r_{n+1})} \subseteq B(x_n, r_n) \cap A_{n+1}$$

and $r_n \rightarrow 0$ as $n \rightarrow +\infty$.

The construction is inductive.

Consider

$$\overline{B}(x_1, r_1) \subseteq \underbrace{B(x_0, r_0) \cap A_1}_{\text{nonempty open set because } A_1 \text{ is open and dense}}$$

If $x_1, \dots, x_n$ have been defined, then consider

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \underbrace{B(x_n, r_n) \cap A_{n+1}}_{\text{open set} \neq \emptyset}$$

wlog we can assume that $r_n \leq \frac{1}{n}$.

The sequence $\{x_n\}$ is a Cauchy sequence because if $n \geq n_0$ and $m \geq n_0$, then

$$\text{dist}(x_n, x_m) \leq \underbrace{\text{dist}(x_n, x_{n_0})}_{\leq r_{n_0}} + \underbrace{\text{dist}(x_m, x_{n_0})}_{\leq r_{n_0}}$$

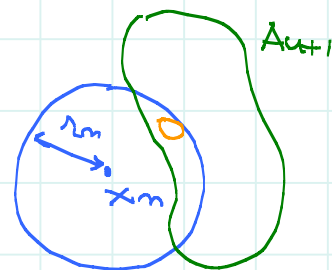
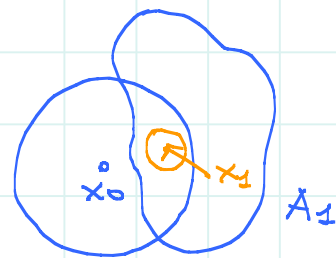
Therefore $x_n \rightarrow x_\infty$. But $x_\infty \in \overline{B}(x_n, r_n)$ for every $n \geq 1$ and therefore

$$x_\infty \in \bigcap_{n=1}^{\infty} A_n \cap \overline{B}(x_0, r_0)$$

This proves that the intersection is dense

Remark The proof of case (2) is analogous, just define a sequence of set $\{U_n\}$ s.t.

$$\underbrace{\text{Clos}(U_{n+1})}_{\text{compact}} \subseteq U_n \cap A_{n+1}$$



Remark The idea for ③ is the following. Let $Y \subseteq X$ be an open set. Let $\{A_i\}_{i \in \mathbb{N}}$ be a seq. of open dense subsets of Y . Then

$$\hat{A}_i := A_i \cup (X \setminus \text{clos}(Y))$$

are open dense subsets of X , and therefore

$$\bigcap_{i=1}^{\infty} \hat{A}_i = \bigcap_{i=1}^{\infty} A_i \cup (X \setminus \text{clos}(Y))$$

is dense in X , but this is possible only if the first intersection is dense in Y .

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Interesting exercise → Prove that a closed subset of a Baire space is a Baire space
→ Find a counterexample (it is false)
→ Realize that the restricted topology is always difficult to understand.

Remark → Any open subset of a Banach space is a Baire space (open set in Baire space)
→ Any closed subset of a Banach space is a Baire space (it is again a complete metric space)
→ Any closed subset of a loc. comp. top. space is a Baire space (it is again a loc. comp. top. space)
→ $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ are Baire spaces (complete metric spaces)
→ $\mathbb{Q}$ is not a Baire space
→ $\mathbb{R} \setminus \mathbb{Q}$ are a Baire space } exercises

BAIRE POINT 1

Let $f: (0, +\infty) \rightarrow \mathbb{R}$ be a continuous function.

Let us assume that for every $x > 0$ there exists

$$\lim_{n \rightarrow +\infty} f(nx) = l \in \mathbb{R} \quad (\text{always the same})$$

Then
$$\lim_{x \rightarrow +\infty} f(x) = l$$

Proof For every $\varepsilon > 0$ we know that

$$|f(nx) - l| \leq \varepsilon \quad \text{for } n \text{ LARGE ENOUGH}$$

QUALITATIVE

Let us introduce

$$C_k := \{x > 0 : |f(nx) - l| \leq \varepsilon \quad \forall n \geq k\}$$

QUANTITATIVE

$$= \bigcap_{n \geq k} \underbrace{\{x > 0 : |f(nx) - l| \leq \varepsilon\}}_{\text{closed subset of } (0, +\infty)}$$

By assumption
$$\bigcup_{k=1}^{\infty} C_k = (0, +\infty)$$

Baire lemma $\Rightarrow \exists k_0$ s.t. C_{k_0} has nonempty interior, namely $(a, b) \subseteq C_{k_0}$ for some interval (a, b) , namely

$$|f(nx) - l| \leq \varepsilon \quad \forall n \geq k_0 \quad \forall x \in (a, b)$$

or equivalently

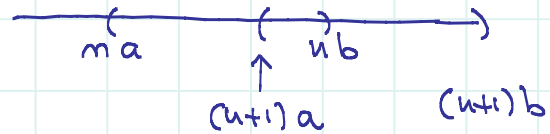
$$|f(y) - l| \leq \varepsilon \quad \forall y \in (na, nb) \quad n \geq k_0$$

we observe that

$$|f(y) - l| \leq \varepsilon$$

$$\forall y \in \bigcup_{n \geq k_0} (na, nb)$$

this union contains a
half-line $(A_0, +\infty)$



The intervals start to overlap as soon as

$$(n+1)a < nb \quad \leadsto \quad a < n(b-a)$$

$\uparrow$ true for n large enough

Nice variant Consider the case when l might depend on x .