

Weak convergence in L^p spaces

Balls in L^p with $p \in (1, +\infty)$ (endpoints excluded) are weakly compact, at least for classical measure spaces.

Proof 1 L^p is reflexive for p in this range

Proof 2 L^p is the dual of $L^{p'}$ where $\frac{1}{p'} + \frac{1}{p} = 1$. The ball in L^p coincides with the ball of $(L^{p'})'$ which is weakly compact.

Finally we observe that weak convergence in $(L^{p'})'$ coincides with weak convergence in L^p .

$$\{f_n\} \subseteq L^p \quad \int f_n g \rightarrow \int f_\infty g \quad \forall g \in L^{p'} \quad (*)$$

Proof 3 Let us work by hands. Take a sequence $\{f_n\} \subseteq L^p$ bounded. Want to prove (*).

Step 1 Up to subsequences (*) is true for every $g \in \mathcal{D} \subseteq L^{p'}$
↑
countable dense subset

Step 2 Up to the same subsequence

$$\int f_n g \rightarrow L(g) \quad \forall g \in L^{p'}$$

Step 3 We know that $L(g) \in (L^{p'})'$. Now we need to know that $(L^{p'})' = L^p$ and this requires the representation theorems.

LAX-MILGRAM Simple version.

For every $f \in L^2((0,1))$ there exists a unique $u \in H_0^1((0,1))$ such that

$$u'' = f$$

in the sense that

$$\int_0^1 u' \varphi' dx = - \int_0^1 f \varphi$$

$$\forall \varphi \in C_c^\infty((0,1)) \quad (*)$$

↑
and also in $H_0^1((0,1))$

Proof 1 Variational formulation

$$\leadsto \min \left\{ \int_0^1 \left[\frac{1}{2} u'^2 + f(x) u \right] dx : u \in H_0^1((0,1)) \right\}$$

Direct method $\Rightarrow$ existence

ELE in the 1st int. form $\rightarrow$
 u satisfies $(*)$.

Proof 2 Define

$$F(x) := \int_0^x f(t) dt \quad \text{and}$$

$$\hat{F}(x) := \int_0^x F(t) dt + ax + b$$

then adjust the constants a and b so that $\hat{F} \in H_0^1((0,1))$.

Proof 3 À la LAX-MILGRAM.

Consider $H := H_0^1((0,1))$. Consider

$$\langle u, v \rangle_H := \int_0^1 u' v' dx$$

This is a scalar product:

$\rightarrow$ bilinear is trivial

$\rightarrow$ positive definite ok due to boundary conditions

$\rightarrow$ completeness is an exercise. If $\{u_n\}$ is a Cauchy seq. in H then $\{u_n'\}$ is a Cauchy seq. in $L^2((0,1))$ and

$\{u_n\}$ is bounded in $L^\infty((0,1))$. $\Leftarrow$ again we use DBC

At this point $\dot{u}_n \rightarrow v_0$ strongly in L^2 and $u_n \rightarrow u_0$ uniformly (at least up to subsequences).

At this point $v_0 = \dot{u}_0$ and by the sub-sub lemma the whole sequence u_n converges.

Now consider $L: H \rightarrow \mathbb{R}$ defined by

$$L(v) := - \int_0^1 f(x) v(x) dx$$

Clearly L is linear. Is L continuous in H , we need

$$|L(v)| \leq \|f\|_{L^2((0,1))} \cdot \|v\|_{L^2((0,1))}$$

$$\begin{aligned} &\leq \|f\|_{L^2((0,1))} \cdot \text{const} \cdot \underbrace{\|\dot{v}\|_{L^2((0,1))}}_{\|v\|_H} \\ &\quad \uparrow \\ &\text{Poincaré} \end{aligned}$$

Therefore $L \in H'$ and any element of H' can be represented as a scalar product, namely there exists $u \in H$ s.t.

$$\begin{aligned} L(v) &= \langle u, v \rangle_H \\ &= - \int_0^1 f v = \int_0^1 u \dot{v} \quad \forall v \in H_0^1((0,1)) \end{aligned}$$

as required

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Example Strange elements in $[L^\infty((-1,1))]'$

POINTWISE VALUES

$$C^0([-1,1]) \ni u(x) \longrightarrow u(0) \in \mathbb{R}$$

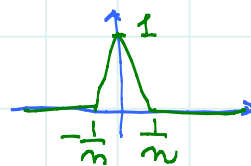
$\uparrow$ vector subspace of $L^\infty((-1,1))$ " $L(u)$ "

The linear map defined above is cont. wrt the norm of L^∞ .
HB $\Rightarrow$ we can extend L to a lin. cont. function on $L^\infty((-1,1))!$

It is possible to see that there is no $g \in L^1((-1,1))$ s.t.

$$\int_{-1}^1 g(x) u(x) dx = u(0) \quad \forall u \in C^1([-1,1])$$

[Consider $u_n(x)$ as in the figure]



Remark If we consider the standard proof that works for $p < +\infty$ we define, given $L \in (L^\infty)'$, the measure

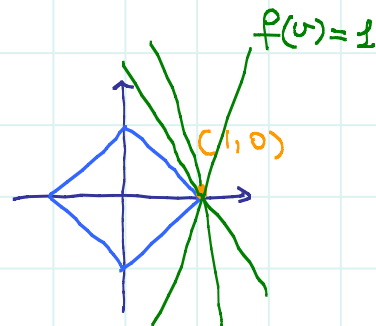
$$\nu(A) := L(1_A) \quad \forall A \text{ measurable}$$

It is possible to check that ν is FINITELY ADDITIVE but NOT countably additive. This is NOT enough for Radon-Nikodym.

Example Non-uniqueness of the aligned functional.

Consider $V = \mathbb{R}^2$ with the 1-norm

$$\|(x,y)\|_1 := |x| + |y|$$



Consider $v = [1, 0]$. Which are the aligned functionals of v ?

They are of the form $f(x,y) = ax + by$

What is $\|f\|_V$? We have to compute

$$\max_{\sup} \{ |\varphi(x, y)| : \|(x, y)\|_1 \leq 1 \} = \max \{ |a|, |b| \}$$

$$\begin{aligned} |f(x,y)| &\leq |a| \cdot |x| + |b| \cdot |y| \\ &\leq \max\{|a|, |b|\} \cdot (|x| + |y|) \\ &= \|f\|_1 \cdot \|(x,y)\|_1 \end{aligned}$$

Given a and b , we can choose x and y in such a way that we have equalities.

So we need that

$$\rightarrow \max \{ |a|, |b| \} \leq 1 \quad \|f\|_{V_1} \leq 1$$

$$\rightarrow a = 1 \quad \varphi(1,0) = \| (1,0) \|_y = 1$$

There are infinitely many possibilities! $\varphi(x,y) = x + by$
with $|b| \leq 1$

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