

REFLEXIVE SPACES

Defn Let V be a normed space. The CANONICAL INJECTION is the map

$$J : V \rightarrow (V')'$$

defined by

Duality pairing between V and V'

$$[J(v)](f) := f(v) \quad \forall v \in V \quad \forall f \in V'$$

Prop. J is a linear ISOMETRY (and in particular injective).

Proof. Linear is trivial. We claim that

$$\|J(v)\|_{(V')'} = \|v\|_V \quad \forall v \in V$$

$$\boxed{\leq} \quad |[J(v)](f)| = |f(v)| \leq \|v\|_V \cdot \|f\|_{V'}$$

Consider the sup of LHS as f varies in $B_{V'}(0,1)$.

$\boxed{\geq}$ For every $v \in V$ there exists $f \in V'$ with $\|f\|_{V'} \leq 1$ such that

$$\|v\|_V = |[J(v)](f)| \leq \|J(v)\|_{(V')'} \cdot \underbrace{\|f\|_{V'}}_1$$

This f is the aligned functional.

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Defn The space V is called **REFLEXIVE** if J is surjective

Prop If V is reflexive, then V is isometric and isomorphic to $(V')'$, and in particular is a Banach space.

Achtung! There are weird examples of spaces where V is isometric and isomorphic to $(V')'$, but the isomorphism is NOT the canonical injection. These spaces are NOT reflexive.

Theorem (Weak compactness of balls in reflexive spaces)

Let V be a normed space, and let $\{v_n\}$ be a sequence in V .

Let us assume that

(i) $\{v_n\}$ is bounded

(ii) V is reflexive

(iii) V' is separable (the price to pay in order to use sequences)

Then there exists

$$v_{n_k} \longrightarrow v_\infty \text{ weakly in } V.$$

Proof Usual argument

Step 1 Let us satisfy a countable dense subset $\{f_i\}$ of V' .

More precisely, there exists v_{n_k} such that

$$\lim_{k \rightarrow \infty} f_i(v_{n_k}) = \text{something} \quad \text{for every } i \in \mathbb{N}$$

[Usual diagonal argument after observing that

$$|f_i(v_n)| \leq \underbrace{\|f_i\|_{V'}}_{\text{fixed}} \cdot \underbrace{\|v_n\|}_{\text{bounded}}$$

Step 2 We show that

$$\lim_{k \rightarrow \infty} f(u_{m_k}) = \text{something} \quad \forall f \in V'$$

$\uparrow$
the SAME subsequence that
satisfies $\{f_m\}_{m \geq 1}$

[We use that $\{f(u_{m_k})\}$ is a Cauchy sequence]

Step 3 Let $L(f)$ denote the limit that we found in Step 2. It is possible to see that

- L is linear
- $L: V' \rightarrow \mathbb{R}$ is continuous

$$|L(f)| = \lim_{k \rightarrow \infty} |f(u_{m_k})| \leq \|f\|_{V'} \cdot \underbrace{\|u_{m_k}\|_V}_{\text{bounded}}$$

This means that $L \in (V')'$ and therefore $L = J(u_\infty)$ for some $u_\infty \in V$.

This proves that

$$\lim_{k \rightarrow \infty} f(u_{m_k}) = L(f) = [J(u_\infty)](f) = f(u_\infty) \quad \therefore$$

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Corollary Assume that V is a reflexive space with V' separable. Then for every $f \in V'$ it turns out that

$$\|f\|_{V'} = \max \{ |f(v)| : v \in V \text{ and } \|v\|_V \leq 1 \}$$

$\uparrow$
and not only sup

Proof Let $\{u_m\}$ be a sep-izing sequence. It admits a weakly conv. subsequence $u_{m_k} \rightharpoonup u_\infty$, and finally

$$f(u_\infty) = \lim_{k \rightarrow \infty} f(u_{m_k}) = \|f\|_{V'}$$

$\uparrow$ weak conv. $\uparrow$ definition of u_m

Added after video: proof without assuming V' to be separable. Consider the aligned functional of f in $(V')'$. It is $J(\text{something})$

Remark V separable $\not\Rightarrow V'$ separable

[For example L^1 and ℓ^1 are separable, but L^∞ and ℓ^∞ are not separable]

Prop. (The converse is true: V' separable $\Rightarrow V$ is separable)
We assume that V is a normed space.

Proof Step 1 V' separable $\Rightarrow$ unit sphere of V' is separable
(any subset of a separable metric space is separable, because a metric space is sep. $\Leftrightarrow$ it admits a countable basis)

Let $\{f_m\}_{m \geq 1}$ be a countable dense subset in this sphere.

Since $\|f_m\|_{V'} = 1$, there exists $u_m \in V$ such that

$$\|u_m\|_V \leq 1 \quad \text{and} \quad f(u_m) \geq \frac{1}{2} \quad \uparrow \text{any constant } < 1 \text{ is good}$$

Let $W := \text{Clos}(\text{Span}(u_1, \dots, u_m, \dots))$. We claim that $W = V$.

[If YES, then the FINITE lin. comb. of $u_1, \dots, u_m, \dots$ with coeff. in $\mathbb{Q}$ are a countable dense subset of V].

Step 2 We prove the claim. Assume by contrad. it is false.

Then there exists $z \in V \setminus W$ and $\|z\|_V = 1$.

Now there exists $g \in V'$ such that

$$\begin{aligned} g(z) &= 1 \\ g(w) &= 0 \quad \forall w \in W \end{aligned}$$

[Existence follows from Hahn-Banach]

Note that the extension satisfies $\|g\|_{V'} = 1$

Added after video: this is not that simple.
Please check the last page of the lecture.

But now $g \in$ unit sphere in V' and therefore there exists $n \geq 1$ such

$$\|g - f_n\|_{V'} \leq \frac{1}{4}$$

On the other hand

$$\frac{1}{2} \leq f_n(v_n) = f_n(v_n) - g(v_n) + g(v_n)$$

" because $v_n \in W$

$$\leq \|f_n - g\|_{V'} \cdot \|v_n\|_V$$

$\leq \frac{1}{4}$ ≤ 1

$$\leq \frac{1}{4}$$

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THE PARADOX OF HILBERT TRIPLES

Consider $H := L^2((0,1))$ and $V := H^1((0,1))$.

They are Hilbert spaces and $V \subseteq H$.

Consider $f \in H'$. So $f: H \rightarrow \mathbb{R}$ is linear and cont. in H .

The restriction of f to V is linear and cont. w.r.t to V because

$$|f(v)| \leq \text{const} \cdot \|v\|_H \leq \text{const} \cdot \|v\|_V$$

This means that there exist a map

$$H' \longrightarrow V'$$

which is injective (if $f(v) = 0$ for every $v \in V$, then $f(v) = 0$ for every $v \in H$ because V is dense).

In other words

$$V \subseteq H = H' \subseteq V' = V$$

$\uparrow$ $\uparrow$
 dual of Hilbert spaces
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[This part was added after video]

General fact V normed space, $W \subseteq V$ CLOSED subspace with $W \neq V$.

Then there exists $g \in V'$ such that $\|g\|_{V'} = 1$ and $g(w) = 0$ for every $w \in W$.

Proof Let us assume that $B(v_0, r_0) \subseteq V \setminus W$. It is enough to find $g \in V'$ with $g(v_0) = 1$ and $g(w) = 0$ for every $w \in W$. This function g does not satisfy necessarily $\|g\|_{V'} = 1$, but a suitable multiple does.

Consider $E := W \oplus \mathbb{R}v_0$, and define $g: E \rightarrow \mathbb{R}$ by

$$g(w + \alpha v_0) = \alpha \quad \forall \alpha \in \mathbb{R} \quad \forall w \in W.$$

Claim: this function satisfies

$$g(w + \alpha v_0) \leq \frac{1}{r_0} \|w + \alpha v_0\| \quad \forall \alpha \in \mathbb{R} \quad \forall w \in W.$$

and therefore HB provides an extension with the same property.

Indeed (wlog $\alpha > 0$):

$$\begin{aligned} g(w + \alpha v_0) &= \alpha = \alpha \frac{r_0}{r_0} \leq \frac{1}{r_0} \alpha \operatorname{dist}\left(v_0, \underbrace{-\frac{w}{\alpha}}_{\substack{\text{this is in } W, \text{ and hence not} \\ \text{in } B(v_0, r_0)}}\right) \\ &= \frac{1}{r_0} \alpha \left\| v_0 + \frac{w}{\alpha} \right\| \\ &= \frac{1}{r_0} \|\alpha v_0 + w\| \end{aligned}$$

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