

Dual of L^p spaces

Setting : $(X, \mathcal{m}, \mu)$ is a measure space, namely

- X is a set
- $\mathcal{m}$ is a σ -algebra of subsets
- $\mu: \mathcal{m} \rightarrow [0, +\infty]$ is a measure (countably additive on disjoint subsets)

$L^p(X, \mathcal{m}, \mu)$ usual L^p space with $p \in [1, +\infty]$.

Model case : when the measure is finite $\mu(X) < +\infty$

RADON-NIKODYM THEOREM Let μ be a finite measure on X .

Let $\nu: \mathcal{m} \rightarrow \mathbb{R}$ be a signed measure (count. additive)

Let us assume that ν is absolutely continuous wrt μ , namely

$$\forall A \in \mathcal{m} \quad \mu(A) = 0 \Rightarrow \nu(A) = 0.$$

Then there exists $g \in L^1(X, \mathcal{m}, \mu)$ such that

$$\nu(A) = \int_A g(x) d\mu \quad \forall A \in \mathcal{m}.$$

DUALITY PAIRING BETWEEN FUNCTIONS

$$[J(g)](f) := \int_X g(x) f(x) d\mu$$

well-defined under suitable assumptions on f and g .

Quick statement: the dual of L^p is $L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$
if $p \in [1, +\infty)$ ↑↑ EXCLUDED

In the sequel we assume that $\mu(X) < +\infty$.

Fact 1 For every $p \in [1, +\infty]$ the duality pairing defines a map

$$J: L^{p'} \rightarrow (L^p)'$$
$$g \mapsto J(g)$$

that is linear, well-defined, and continuous

Proof Linearity is clear.

$$\begin{array}{c} | [J(g)](f) | \\ \uparrow \quad \uparrow \\ L^{p'} \quad L^p \end{array} = \left| \int_X f(x)g(x) d\mu \right| \leq \|g\|_{L^{p'}} \cdot \|f\|_{L^p}$$

↑
Hölder ineq.

This proves that $J(g)$ is well-defined as an element of $(L^p)'$ and the norm satisfies

$$\|J(g)\|_{(L^p)'} \leq \|g\|_{L^{p'}}$$

↑
sup of LHS among all f 's in L^p with $\|f\|_{L^p} \leq 1$

Remark We did not exploit that $\mu(X) < +\infty$ and the argument works even for $p=1$ and $p=+\infty$.

Fact 2 For every $p \in [1, +\infty]$, the duality pairing is an isometry namely

$$\|J(g)\|_{(L^p)'} = \|g\|_{L^{p'}}$$

Proof We distinguish 3 cases

$p = +\infty$ Given $g \in L^1$ we claim that $\|J(g)\|_{(L^\infty)'} \geq \|g\|_{L^1}$
We define

$$f(x) := \operatorname{sign}(g(x)) \in L^\infty(\mathbb{X})$$

and observe that

$$\begin{aligned} \|g\|_{L^1} &= \int_{\mathbb{X}} |g(x)| dx = \int_{\mathbb{X}} g(x) \operatorname{sign}(g(x)) \\ &= \int_{\mathbb{X}} g(x) f(x) dx \\ &= [J(g)](f) \\ &\leq \|J(g)\|_{(L^\infty)'} \|f\|_{L^\infty} \\ &\leq \|J(g)\|_{(L^\infty)'} \end{aligned}$$

$p = 1$ Given $g \in L^\infty(\mathbb{X})$ we claim that $\|J(g)\|_{(L^1)'} \geq \|g\|_{L^\infty}$

For every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subseteq \mathbb{X}$ s.t.

$$|g(x)| \geq \|g\|_{L^\infty} - \varepsilon \quad \forall x \in A_\varepsilon \quad \text{and} \quad \operatorname{meas}(A_\varepsilon) > 0$$

w.l.o.g. we can assume $g(x) \geq \|g\|_{L^\infty} - \varepsilon$ (the other case is analogous)

Now we observe that

$$\begin{aligned} (\|g\|_{L^\infty} - \varepsilon) \operatorname{meas}(A_\varepsilon) &\leq \int_{A_\varepsilon} g(x) dx = \int_{\mathbb{X}} g(x) \mathbb{1}_{A_\varepsilon}(x) d\mu \\ &= [J(g)](\mathbb{1}_{A_\varepsilon}) \leq \|J(g)\|_{(L^1)'} \cdot \|\mathbb{1}_{A_\varepsilon}\|_{L^1} = \|J(g)\|_{(L^1)'} \operatorname{meas}(A_\varepsilon) \end{aligned}$$

Now it is enough to simply mean $(A\varepsilon)$ and then let $\varepsilon \rightarrow 0^+$.

$p \in (1, +\infty)$ Given $g \in L^p$ we claim that $\|J(g)\|_{(L^p)'} \geq \|g\|_{L^p}$

We define $f(x) := |g(x)|^{p'-1} \operatorname{sign}(g(x))$

We need that $f \in L^p$ but, as in the case of sequences,

$$\int_{\mathbb{X}} |f(x)|^p d\mu = \int_{\mathbb{X}} |g(x)|^{p(p'-1)} d\mu = \int_{\mathbb{X}} |g(x)|^{p'} d\mu$$

$\uparrow$
 $p(p'-1) = p'$

Now we observe that

$$\begin{aligned} \int_{\mathbb{X}} |g(x)|^{p'} d\mu &= \int_{\mathbb{X}} g(x) f(x) d\mu = [J(g)](f) \\ &\leq \|J(g)\|_{(L^p)'} \cdot \|f\|_{L^p} \\ &\leq \|J(g)\|_{(L^p)'} \left\{ \int_{\mathbb{X}} |g(x)|^{p'} d\mu \right\}^{\frac{1}{p}} \end{aligned}$$

If we simplify we obtain the conclusion

Remark Again we did NOT use that $\mu(\mathbb{X}) < +\infty$ and the argument works also for $p=1$ and $p=+\infty$.

Fact 3 If $p \in [1, +\infty)$, then the duality pairing is surjective.

Let $L: L^p \rightarrow \mathbb{R}$ be a continuous linear functional.

We need to find $g \in L^{p'}$ s.t.

$$L(f) = [J(g)](f) \quad \forall f \in L^p$$

3.1 let us set

$$v(A) := L(\mathbb{1}_A) \quad \forall A \in \mathcal{M}$$

$\in L^p$ because $\text{meas}(A) < +\infty$

Claim: v is a signed measure in $\mathbb{X}$ and v is abs. cont. wrt μ

$$\mu(A) = 0 \Rightarrow v(A) = 0 \rightsquigarrow \text{trivial}$$

$$v(\emptyset) = 0 \rightsquigarrow \text{trivial}$$

The main point is countable additivity. Let $\{A_k\}_{k \geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{M}$.

Let

$$A_\infty := \bigcup_{i \geq 1} A_i.$$

Then we need that $v(A_\infty) = \sum_{i=1}^{\infty} v(A_i)$. But

$$v(A_\infty) = L(\mathbb{1}_{A_\infty})$$

$$= \lim_{n \rightarrow +\infty} L\left(\mathbb{1}_{\bigcup_{i=1}^n A_i}\right)$$

Here we need that

$$\mathbb{1}_{\bigcup_{i=1}^n A_i} \longrightarrow \mathbb{1}_{A_\infty} \text{ in } L^p(\mathbb{X})$$

and this is true only
if $p < +\infty$

A_i 's disjoint $\nearrow$

$$= \lim_{n \rightarrow +\infty} L\left(\sum_{i=1}^n \mathbb{1}_{A_i}\right)$$

$$= \lim_{n \rightarrow +\infty} \sum_{i=1}^n L(\mathbb{1}_{A_i})$$

$$= \lim_{n \rightarrow +\infty} \sum_{i=1}^n v(A_i)$$

$$= \sum_{i=1}^{\infty} v(A_i)$$

3.2 By RN there exists $g \in L^1$ such that

$$v(A) = L(\mathbb{1}_A) = \int_A g(x) d\mu = \int_{\mathbb{X}} g(x) \mathbb{1}_A(x) d\mu$$

The claim is that $g \in L^{p'}$.

At this point L and $J(g)$ are two elements of $(L^p)'$ that coincide on $\mathbb{1}_A$, therefore also on the closure of the span which is L^p if $p < +\infty$.

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