

TOUR IN THE DUAL OF ℓ^∞

Let us consider the duality pairing $[J(y)](x) := \sum_{i=1}^{\infty} x_i y_i$

Step 1 It defines a map $J: \ell^1 \rightarrow (\ell^\infty)'$ that is well-defined, linear and continuous

$$|[J(y)](x)| \leq \sum_{i=1}^{\infty} |x_i| \cdot |y_i| \leq \|y\|_{\ell^1} \|x\|_{\ell^\infty}$$

which proves it is well-defined and $\|J(y)\|_{(\ell^\infty)'} \leq \|y\|_{\ell^1}$

Step 2 J is an ISOMETRY, and therefore injective

Given $y \in \ell^1$, define $x \in \ell^\infty$ with $x_i := \text{sign}(y_i)$. Then

$$\|y\|_{\ell^1} = \sum_{i=1}^{\infty} |y_i| = \sum_{i=1}^{\infty} x_i y_i = [J(y)](x) \leq \|J(y)\|_{(\ell^\infty)'} \cdot \underbrace{\|x\|_{\ell^\infty}}_1$$

[one should discuss $y=0$ separately]

Step 3 let's try to prove surjectivity

Given $L \in (\ell^\infty)'$, let us define y by $y_i := L(e_i)$ (rec. cond.)

Claim: $y \in \ell^1$ and $J(y) = L$.

$$\begin{aligned} \sum_{i=1}^N |y_i| &= \sum_{i=1}^N \underbrace{\text{sign}(y_i)}_{z_i} \underbrace{y_i}_{L(e_i)} = \sum_{i=1}^N z_i L(e_i) = L\left(\sum_{i=1}^N z_i e_i\right) \\ &\leq \|L\|_{(\ell^\infty)'} \cdot \underbrace{\left\|\sum_{i=1}^N z_i e_i\right\|_{\ell^\infty}}_{\leq 1} \end{aligned}$$

So we have proved that $y \in \ell^1$.

As in the previous cases we know that L and $J(y)$ coincide in $\text{Clos}(\text{Span}(e_i))$, which is NOT ℓ^∞ , BUT c_0 .

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Actually we have almost proved the following

MISLEADING THEOREM The duality pairing defines a map
 $J: \ell^1 \rightarrow (c_0)'$

with the usual three properties

We need to modify the argument in step 2, because we need $x \in c_0$ and not in ℓ^∞ .

Given $y \in \ell^1$ and $N \geq 1$ define

$$(x_N)_i := \begin{cases} \text{sign}(y_i) & \text{if } i \leq N \\ 0 & \text{if } i \geq N+1 \end{cases}$$

Then

$$\sum_{i=1}^N |y_i| = \sum_{i=1}^N y_i (x_N)_i = [J(y)](x_N) \leq \|J(y)\|_{(c_0)'} \|x_N\|_{c_0} \leq 1$$

Letting $N \rightarrow \infty$ we obtain $\|J(y)\|_{(c_0)'} \geq \|y\|_{\ell^1}$

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Prop. $c_{00}' = c_0' = \ell^1$

Proof Consider any element of c_{00}' . It is a linear cont. funct.
 $L: c_{00} \rightarrow \mathbb{R}$

Cont. $\Rightarrow$ Lip $\Rightarrow$ unif. cont. $\Rightarrow$ it can be extended in a unique way to $\hat{L}: c_0 \rightarrow \mathbb{R}$ which remains lip. and linear.

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Very brutal statement: $(C)' = \ell^1$

Less brutal statement: $(C)' = \ell^1 \oplus_{\pm} \mathbb{R} \sim \ell^1$

$\uparrow$ set of pairs (y, z) with $y \in \ell^1$ and $z \in \mathbb{R}$
with $\|(y, z)\| = \|y\|_{\ell^1} + |z|$

$$(y, z) \rightsquigarrow (z, y_1, y_2, y_3, \dots)$$

Theorem The duality pairing

$$[\underbrace{J(y, y_{\infty})}_{\substack{\uparrow \\ \ell^1}}](x) := \sum_{i=1}^{\infty} x_i y_i + y_{\infty} \cdot \underbrace{x_{\infty}}_{\substack{\uparrow \\ \lim_{i \rightarrow \infty} x_i}}$$

defines a map $J : \ell^1 \oplus_{\pm} \mathbb{R} \rightarrow C'$ with the usual three properties

Proof **Step 1** As usual we obtain $\|J(y, y_{\infty})\|_{C'} \leq \|y\|_{\ell^1} + |y_{\infty}|$

Step 2 Given $y \in \ell^1 \oplus_{\pm} \mathbb{R}$ define x_N as

$$(x_N)_i := \begin{cases} \text{sign}(y_i) & \text{if } i \leq N \\ \text{sign}(y_{\infty}) & \text{if } i \geq N+1 \end{cases}$$

With this choice as before we obtain the opposite inequality
(please check the details)

Step 3 Given $L \in C'$ we need to find $(y, y_{\infty}) \in \ell^1 \oplus_{\pm} \mathbb{R}$
such that $L = J(y, y_{\infty})$

To this end we set $y_i := L(e_i)$
 $y_{\infty} := L(\underbrace{e_{\infty}}_{(1, 1, 1, \dots)}) - \sum_{i=1}^{\infty} L(e_i)$

As in the ℓ^∞ case we can prove that $y \in \ell^1$, and therefore also y_∞ is well-defined.

Finally we have to check that L and $J(y, y_\infty)$ coincide in C .

→ For sure they coincide in e_i

→ They coincide if $x = e_\infty$

We observe that e_i 's and e_∞ span a dense subset of C .

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WHAT ABOUT THE DUAL OF ℓ^∞

It contains $J(y)$ for every $y \in \ell^1$, and more precisely we proved that $J: \ell^1 \rightarrow (\ell^\infty)'$ is an injective ISOMETRY.

BUT IT IS NOT SURJECTIVE

Let us find $L \in (\ell^\infty)'$ which is NOT in the image of J .

Consider $L: C \rightarrow \mathbb{R}$ defined as

$$L(x) := \lim_{i \rightarrow \infty} x_i$$

This map is linear and continuous wrt the norm of ℓ^∞

Thanks to HB we can extend L to some $\hat{L}: \ell^\infty \rightarrow \mathbb{R}$.

Fact 1 $\hat{L}$ cannot be represented as $J(y)$ for some $y \in \ell^1$.

Assume that $y_i \neq 0$ for some $i \geq 1$. If x and $\hat{x}$ coincide everywhere but not in position i , then $[J(y)](x) \neq [J(y)](\hat{x})$

On the other hand, both x and $\hat{x}$ can have the same limit at ∞ , and therefore the same $\hat{L}$.

Alternative proof: consider $\hat{L}(e_i)$.

Fact 2 $\hat{L}$ is NOT the liminf or limsup (they are NOT linear)

Fact 3 For every $x \in \ell^\infty$ it turns out that

$$\liminf_{i \rightarrow \infty} x_i \leq \hat{L}(x) \leq \limsup_{i \rightarrow \infty} x_i$$

For example consider $x_0 = 7, 9, 7, 9, 7, 9, \dots$

From the proof of HB we know that

$$L(z) - \|z - x_0\| \leq \hat{L}(x_0) \leq \|y + x_0\|_{\ell^\infty} - L(y) \quad \begin{array}{l} \forall y \in C \\ \forall z \in C \end{array}$$

Consider $y = 0, 0, 0, \dots \rightsquigarrow \hat{L}(x_0) \leq 9$

Consider $z = 9, 9, 9, 9, \dots \rightsquigarrow \hat{L}(x_0) \geq 9 - 2 = 7$

Fact 4 $\forall \alpha \in [7, 9]$ there exists an extension $\hat{L}$ with

$$\hat{L}(x_0) = \alpha \quad !!$$

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