

SEQUENCE SPACESNotation $x = \{x_i\}_{i \geq 1}$ seq. of real numbers.Definition

- We say that $x \in \ell^p$ with $p \in [1, +\infty)$ if

$$\|x\|_{\ell^p}^p := \sum_{i=1}^{\infty} |x_i|^p < +\infty$$

- We say that $x \in \ell^\infty$ if

$$\|x\|_{\ell^\infty} := \sup \{ |x_i| : i \geq 1 \} < +\infty$$

- we say that $x \in c$ if $\lim_{i \rightarrow +\infty} x_i$ exists

- we say that $x \in c_0$ if $\lim_{i \rightarrow +\infty} x_i = 0$

- we say that $x \in c_{00}$ if $x_i = 0$ eventually (for i large enough)

We use the norm ℓ^∞ also in c, c_0, c_{00} .

Remark Of course

$$c_{00} \subseteq c_0 \subseteq c \subseteq \ell^\infty$$

Remark The spaces ℓ^p with $p \in [1, +\infty]$, c and c_0 are Banach spaces

The space c_{00} is NOT complete, and its completion is c_0

[Proofs are an exercise: in the case of ℓ^p one can imitate the elementary proof in the case ℓ^2 , in the case of c and c_0 it is the usual result that limits commute with uniform convergence]

DUALITY PAIRING

Given two sequences x and y we define

$$[J(y)](x) := \sum_{i=1}^{\infty} x_i y_i$$

The series converges under suitable assumptions
— o — o —

Topological dual of ℓ^1

Brutal statement: $(\ell^1)' = \ell^\infty$ (the top. dual of ℓ^1 is ℓ^∞)

Theorem The duality pairing induces a map

$$J: \ell^\infty \rightarrow (\ell^1)'$$

which is

- well-defined, linear, continuous
- an ISOMETRY (and hence injective)
- surjective.

Proof For every $y \in \ell^\infty$ we consider

$$x \mapsto [J(y)](x) = \sum_{i=1}^{\infty} x_i y_i \quad \forall x \in \ell^1$$

$$\text{Step 1} \quad \sum_{i=1}^{\infty} |x_i y_i| \leq \|y\|_{\ell^\infty} \sum_{i=1}^{\infty} |x_i| = \|y\|_{\ell^\infty} \|x\|_{\ell^1}$$

This proves that $J(y) \in (\ell^1)'$ because it is clearly linear.

The same inequality proves that

$$\|J(y)\|_{(\ell^1)'} \leq \|y\|_{\ell^\infty}$$

because

$$\|J(y)\|_{(\ell^1)'} := \sup \{ |[J(y)](x)| : x \in \ell^1, \|x\|_{\ell^1} \leq 1 \}$$

Step 2 We claim that $\|J(y)\|_{(\ell^1)'} = \|y\|_{\ell^\infty}$

By definition of norm in ℓ^∞ , for every $\varepsilon > 0$ there exists $k \geq 1$ such that

$$|y_k| \geq \|y\|_{\ell^\infty} - \varepsilon$$

Define

$$x_i := \begin{cases} \text{sign}(y_k) & \text{if } i=k \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\|y\|_{\ell^\infty} - \varepsilon \leq |y_k| = y_k \cdot \text{sign}(y_k) = [J(y)](x) \leq \|J(y)\|_{(\ell^1)'} \underbrace{\|x\|_{\ell^1}}_1$$

Since ε is arbitrary, this proves that $\|y\|_{\ell^\infty} \leq \|J(y)\|_{(\ell^1)'}$

Step 3 We claim that J is surjective, namely for every $L \in (\ell^1)'$ there exists $y \in \ell^\infty$ such that

$$L(x) = [J(y)](x) \quad \forall x \in \ell^1 \quad (*)$$

In order to define y we set $y_i := L(e_i)$

$\uparrow$
 $0, \dots, 0, \underset{\substack{\uparrow \\ \text{pos } i}}{1}, 0, \dots, 0, \dots$

Let us check that $y \in \ell^\infty$:

$$|y_i| \leq \|L\|_{(\ell^1)'} \cdot \underbrace{\|e_i\|_{\ell^1}}_1 \quad \forall i \geq 1$$

Let us check (*). Define

$$\hat{L}(x) := [J(y)](x) \quad \forall x \in \ell^1$$

Observe that L and $\hat{L}$ are elements of $(\ell^1)'$, and coincide if $x = e_i \rightarrow$ they coincide on $\text{Clos}(\text{Span}(e_i))$, but this span is dense in ℓ^1

— o — o —

Theorem (The dual of ℓ^p is $\ell^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ if $p \in (1, \infty)$)

The duality pairing defines a map $J: \ell^{p'} \rightarrow (\ell^p)'$ with the usual three properties

Proof Step 1 J is well-defined and continuous because

$$\begin{array}{c} | [J(y)](x) | \\ \uparrow \quad \uparrow \\ \ell^{p'} \quad \ell^p \end{array} \leq \sum_{i=1}^{\infty} |x_i| \cdot |y_i| \leq \|y\|_{\ell^{p'}} \cdot \|x\|_{\ell^p}$$

$\frac{1}{p} \quad \frac{1}{p'}$

If we take the sup of LHS among all $x \in \ell^p$ with $\|x\|_{\ell^p} \leq 1$ we obtain

$$\|J(y)\|_{(\ell^p)'} \leq \|y\|_{\ell^{p'}}$$

Step 2 We obtain the opposite inequality. Given $y \in \ell^{p'}$ define x by

$$x_i := |y_i|^{p'-1} \text{sign}(y_i)$$

First claim : $x \in \ell^p$

$$\sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{\infty} |y_i|^{p'p-p} = \sum_{i=1}^{\infty} |y_i|^{p'} = \|y\|_{\ell^{p'}}^{p'}$$

$$p'p - p \stackrel{?}{=} p' \Leftrightarrow p'p = p + p' \Leftrightarrow 1 = \frac{1}{p'} + \frac{1}{p}$$

Second claim : $\|J(y)\|_{(\mathcal{Q}^p)'} = \|y\|_{\mathcal{Q}^p}$

$$\begin{aligned}\|y\|_{\mathcal{Q}^p}'^{p'} &= \sum_{i=1}^{\infty} |y_i|^{p'} = \sum_{i=1}^{\infty} x_i y_i = [J(y)](x) \\ &\leq \|J(y)\|_{(\mathcal{Q}^p)'} \|x\|_{\mathcal{Q}^p} \\ &= \|J(y)\|_{(\mathcal{Q}^p)'} \|y\|_{\mathcal{Q}^p}'^{p'/p}\end{aligned}$$

If we simplify $\|y\|_{\mathcal{Q}^p}'^{p'/p}$ we obtain the required inequality

Step 3 $J : \mathcal{Q}^p \rightarrow (\mathcal{Q}^p)'$ is surjective

Take any $L \in (\mathcal{Q}^p)'$ and define $y_i := L(e_i)$ (nec. cond.)

We need to prove that $y \in \mathcal{Q}^p$ and $J(y) = L$.

For the first fact we argue as in step 2

$$\begin{aligned}\sum_{i=1}^N |y_i|^{p'} &= \sum_{i=1}^N \underbrace{|y_i|^{p'-1}}_{z_i} \underbrace{\text{sign}(y_i) y_i}_{L(e_i)} = \sum_{i=1}^N z_i L(e_i) \\ &= L\left(\sum_{i=1}^N z_i e_i\right) \leq \|L\|_{(\mathcal{Q}^p)'} \cdot \left\{ \sum_{i=1}^N |z_i|^p \right\}^{1/p} \\ &= \|L\|_{(\mathcal{Q}^p)'} \cdot \left\{ \sum_{i=1}^N |y_i|^{p'} \right\}^{1/p}\end{aligned}$$

If we simplify and we let $N \rightarrow +\infty$ we obtain that

$$\|y\|_{\mathcal{Q}^p}' \leq \|L\|_{(\mathcal{Q}^p)'}$$

The conclusion follows as in the previous case: L and $J(y)$ are two elements of $(\mathcal{Q}^p)'$ and they coincide on e_i for every $i \geq 1$. Since $p < +\infty$, the span of all e_i 's is dense in $\mathcal{Q}^p$.

— o — o —