

Lemma (PSEUDO-NORM ASSOCIATED TO A CONVEX SET)

Let V be a normed space, and let $C \subseteq V$.

Let us assume that

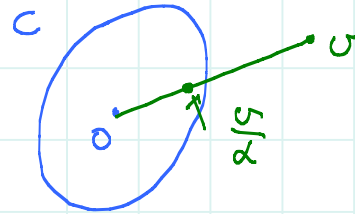
(i) C is convex

(ii) $0 \in C$

(iii) C is open

Let us set

$$p(v) := \inf \left\{ \alpha > 0 : \frac{v}{\alpha} \in C \right\} \quad \forall v \in V$$



Then

(1) p is a pseudo-norm on V

(2) there exists M such that $p(v) \leq M \|v\|_V$

(3) $C = \{v \in V : p(v) < 1\}$ (in some sense C is the unit ball wrt p)

Proof For every $v \in V$ let us set

$$A(v) := \left\{ \alpha > 0 : \frac{v}{\alpha} \in C \right\}$$

so that $p(v) = \inf A(v)$.

(1) $p(\lambda v) = \lambda p(v)$ for every $\lambda > 0$ because $A(\lambda v) = \lambda A(v)$

$p(v_1 + v_2) \leq p(v_1) + p(v_2)$ follows from

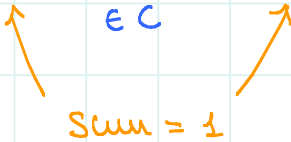
$$A(v_1 + v_2) \supseteq A(v_1) + A(v_2)$$

because

$$\begin{aligned} p(v_1 + v_2) &= \inf A(v_1 + v_2) \leq \inf [A(v_1) + A(v_2)] \\ &= \inf A(v_1) + \inf A(v_2) \\ &= p(v_1) + p(v_2) \end{aligned}$$

In order to prove the inclusion of sets, we consider $\alpha_1 \in A(v_1)$ and $\alpha_2 \in A(v_2)$ and we show that $\alpha_1 + \alpha_2 \in A(v_1 + v_2)$

$$\frac{v_1 + v_2}{\alpha_1 + \alpha_2} = \frac{\alpha_1}{\alpha_1 + \alpha_2} \boxed{\frac{v_1}{\alpha_1}} + \frac{\alpha_2}{\alpha_1 + \alpha_2} \boxed{\frac{v_2}{\alpha_2}} \in C$$



Sum = 1

(2) Since C is open and $0 \in C$, there exists $R > 0$ s.t.

$$\overline{B}_V(0, R) \subseteq C$$

Claim: we can take $R = \frac{1}{p}$. Indeed for every $v \in V$ we have that $\frac{1}{R} \|v\|_V \in A(v)$ because

$$\frac{v}{\frac{1}{R} \|v\|_V} = \text{vector with norm } R \in \overline{B}(0, R) \subseteq C$$

[Trivial also if $\|v\|_V = 0$]

(3) $\boxed{\subseteq}$ Let $v \in C$ then $1 \in A(v) \Rightarrow p(v) < 1$

(in general $A(v)$ is a nonempty open set and it is a right half-line)

$\boxed{\supseteq}$ Let $v \in V$ with $p(v) < 1$. Then $1 \in A(v)$ and therefore $\frac{v}{1} \in C \leadsto v \in C$.

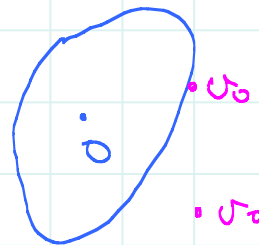
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[Prop.] Let $C \subseteq V$ and let $v_0 \in V$.

Assume that

(i) C is an OPEN convex set and $0 \in C$

(ii) $v_0 \notin C$



Then there exists $f \in V'$ that separates weakly C and v_0

[Proof] We consider $E := \text{Span}(v_0)$ and the function

$$f: E \rightarrow \mathbb{R}$$

defined as

$$f(\alpha v_0) := \alpha \quad \forall \alpha \in \mathbb{R}$$

We observe that f is linear and

$$f(\alpha v_0) \leq p(\alpha v_0) \quad \forall \alpha \in \mathbb{R}$$

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Indeed, the inequality is trivial when $\alpha \leq 0$ and if $\alpha > 0$

it is equivalent to $1 \leq p(v_0)$

and this is true because of (3) of the Lemma.

HB $\Rightarrow$ we can extend f to $f: V \rightarrow \mathbb{R}$.

We claim that this is the required separation.

Indeed

$$f(v) \leq p(v) < 1 = f(v_0) \quad \forall v \in C$$

$\uparrow$ HB $\uparrow$ (3) $\uparrow$ defn

We need to prove that f is continuous w.r.t $\| \cdot \|_V$.

By statement (2) of the previous Lemma we know that

$$f(v) \leq p(v) \leq M \|v\|_V \quad \forall v \in V$$

This is ok if $f(v) \geq 0$. Otherwise

$$|f(v)| = -f(v) = f(-v) \leq p(-v) \leq M \|-v\|_V = M \|v\|_V$$

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Prop. Again $C \subseteq V$ and $v_0 \in V$.

Assume that

(i) C is convex open and nonempty. (now 0 is not nec. in C)

(ii) $v_0 \notin C$

Then there exists a weak separation

Proof Take any $w_0 \in C$. Then consider

$$\hat{C} := C - w_0$$

$$\hat{v}_0 := v_0 - w_0$$

Apply the previous proposition to $\hat{v}_0$ and $\hat{C}$ and simplify $f(w_0)$

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PROOF OF WEAK SEPARATION (1st geom. form of HB)

A, B convex $A \cap B = \emptyset$ A is open $\Rightarrow$ weak sep.

Introduce $C := A - B = \{a - b : a \in A, b \in B\}$

We observe that

$\rightarrow C$ is convex (general fact)

$\rightarrow C$ is open (union of translations of A)

$\rightarrow 0 \notin C$ ($A \cap B = \emptyset$)

By previous prop. we can separate 0 from C , and obtain $f \in V'$ s.t.

$$\begin{array}{ccc} f(c) < f(0) & & \forall c \in C \\ f(a-b) & & 0 \end{array}$$

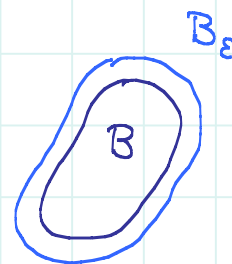
$$\text{namely } \begin{array}{ccc} f(a) - f(b) < 0 & & \forall a \in A \quad \forall b \in B \\ f(a) < f(b) & & \end{array}$$

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PROOF OF STRONG SEPARATION

A, B convex sets, $A \cap B = \emptyset$, A closed and B compact
 $\leadsto$ strong separation

Idea: enlarge one of them a little bit



For every $\varepsilon > 0$ we set $B_\varepsilon := \bigcup_{b \in B} B(b, \varepsilon)$
 $\uparrow$ open ball

Claim: $A \cap B_\varepsilon = \emptyset$ if ε is small enough

Assume it is not the case. Then $A \cap B_{1/n} \neq \emptyset$ for every n
This means that there exist

$$\begin{array}{ccccc} a_n & = & b_n & + & \frac{1}{n} u_n \\ \uparrow & & \uparrow & & \uparrow \\ \in A & & \in B & & \|u_n\|_V \leq 1 \end{array}$$

Up to subsequences $b_n \xrightarrow{\in B} b_\infty$ because B is compact.

But then also $a_n \xrightarrow{\in A} b_\infty$ because A is closed

but $A \cap B = \emptyset$!!

Since B_ε is open, there exists $f \in V'$ such that

$$f(a) < f(b) \quad \forall b \in B_\varepsilon$$

$$f(a) < f(b) + \varepsilon f(u) \quad \forall b \in B \quad \forall u \text{ with } \|u\|_V < 1$$

Now set $\delta_1 := \sup \{ f(a) : a \in A \}$

$$\delta_2 := \delta_1 + \varepsilon \|f\|_{V'}$$

We have $\varphi(a) \leq \delta_1 \quad \forall a \in A$

we claim that

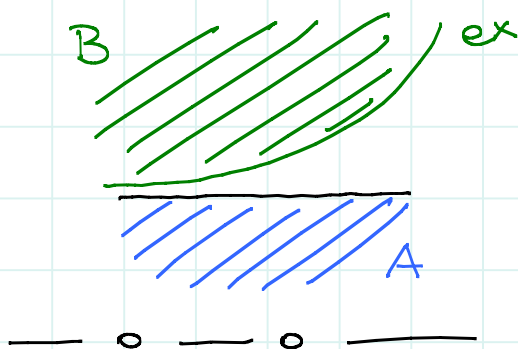
$$\varphi(b) \geq \delta_2 \quad \forall b \in B$$

$$\varphi(b) + \varepsilon \varphi(u) \geq \delta_1 \quad \forall b \in B \quad \forall \|u\|_V < 1$$

$$\leadsto \varphi(b) \geq \delta_1 - \varepsilon \varphi(u) \quad \forall b \in B \quad \forall \|u\|_V < 1$$

If we consider the sup of the RHS among all u 's in the ball we obtain exactly $\varepsilon \|\varphi\|_{V'} + \delta_1$.

Example A and B closed is NOT enough for a strong separation



[Last example edited after video. Summary:

- in finite dimension any two convex disjoint sets admit a weak sep. (nice exercise)
- in infinite dimension there do exist two convex closed disjoint sets that do NOT admit a weak separation (hard exercise)