

Defn Let V be normed space, and let V' be the topological dual.

(1) If $\{u_n\} \subseteq V$ and $u_\infty \in V$, we say that

$$u_n \rightarrow u_\infty \text{ weakly in } V$$

iff

$$f(u_n) \rightarrow f(u_\infty) \quad \forall f \in V'$$

↑
convergence of numbers

(2) If $\{f_n\} \subseteq V'$ and $f_\infty \in V'$, we say that

$$f_n \xrightarrow{*} f_\infty \text{ weakly } * \text{ in } V'$$

iff

$$f_n(v) \rightarrow f_\infty(v) \quad \forall v \in V$$

↑
convergence of numbers

Remark Weak $*$ convergence is actually pointwise conv. of functions.

Remark One day weak conv. will coincide with weak convergence as previously defined in Hilbert spaces.

Basic properties

① Uniqueness of the limit

② Linearity

③ Strong $\Rightarrow$ weak

④ $u_n \rightarrow u_\infty \Rightarrow u_{n_k} \rightarrow u_\infty$

and the same for weak $*$

⑤ Sort of sub-sub-lemma.

Proof of uniqueness Assume that $u_m \rightarrow u_\infty$ and $u_m \rightarrow \hat{u}_\infty$

Then for every $f \in V'$

$$f(u_m) \rightarrow f(u_\infty)$$

$$f(u_m) \rightarrow f(\hat{u}_\infty)$$

$$\text{and hence } f(u_\infty - \hat{u}_\infty) = 0 \quad \forall f \in V'$$

Now use $f =$ aligned functional of $u_\infty - \hat{u}_\infty$ (we used HB and therefore CA)

— o — o —

Theorem (LSC of the norm)

Assume $u_m \rightarrow u_\infty$ weakly in V

$f_m \rightarrow f_\infty$ weakly* in V'

Then

$$\liminf_{m \rightarrow \infty} \|u_m\|_V \geq \|u_\infty\|_V$$

$$\liminf_{m \rightarrow \infty} \|f_m\|_{V'} \geq \|f_\infty\|_{V'}$$

Dim $\square$ By definition $|f(u_m)| \leq \|f\|_{V'} \cdot \|u_m\|_V$
and therefore

$$\begin{aligned} |f(u_\infty)| &= \lim_{m \rightarrow \infty} |f(u_m)| \leq \overbrace{\|f\|_{V'}}^{\leq 1} \cdot \liminf_{m \rightarrow \infty} \|u_m\|_V \\ &\leq \liminf_{m \rightarrow \infty} \|u_m\|_V \end{aligned}$$

provided that $\|f\|_{V'} \leq 1$

Now consider $f =$ aligned functional of u_∞ and obtain $\|u_\infty\|_V$ in the LHS.

$\square$ Observe that $|f_m(u)| \leq \|f_m\|_{V'} \cdot \|u\|_V \leq \|f_m\|_{V'}$
if $\|u\|_V \leq 1$

Therefore for every $v \in V$ with $\|v\|_V \leq 1$ it turns out that

$$|f_\infty(v)| = \lim_{n \rightarrow \infty} |f_n(v)| \leq \liminf_{n \rightarrow \infty} \|f_n\|_{V'}.$$

Now consider the sup of LHS among all such v .
— o — o —

Theorem (Weak * compactness of balls)

Let V be a normed space, and let V' be the top. dual.

Assume V is **SEPARABLE**

Let $\{f_n\} \subseteq V'$ be a **BOUNDED** sequence.

Then there exists

$$f_{n_k} \xrightarrow{*} f_\infty \text{ weakly } * \text{ in } V'.$$

Proof We follow the same road map as in Hilbert spaces.

Let us choose a countable dense set $D \subseteq V$.

Step 1 There exists $n_k \rightarrow \infty$ such that

$$f_{n_k}(v) \text{ has a limit } \forall v \in D$$

Let us $D = \{d_1, d_2, d_3, \dots\}$. Observe that

$$|f_n(d_i)| \leq \underbrace{\|f_n\|_{V'}}_{\leq M} \cdot \|d_i\|_V$$

$\leadsto$ existence of conv. subsequence

Then we consider $d_2 \in D$ and we obtain a further subsequence
... we conclude by the usual diagonal argument

Step 2 We show that $f_{n_k}(v) \rightarrow \text{something } \forall v \in V$
 $\uparrow$
the same subsequence that
is good for D

It is enough to prove that $\{f_{n_k}(v)\}$ is a Cauchy seq. of numbers.

$$|f_{n_k}(v) - f_{n_R}(v)| \leq |f_{n_k}(v) - f_{n_k}(d)| \quad (1)$$

$$+ |f_{n_k}(d) - f_{n_R}(d)| \quad (2)$$

$$+ |f_{n_R}(d) - f_{n_R}(v)| \quad (3)$$

(1) and (3) are small if d is close enough to v , (2) is small if R and k are large enough.

Formally:

→ we are given $\varepsilon > 0$

→ fix $d \in D$ such that $M \|d - v\|_V \leq \frac{1}{4} \varepsilon$ so that

$$|f_n(v) - f_n(d)| \leq \|f_n\|_{V'} \cdot \|v - d\|_V \leq \frac{1}{4} \varepsilon \leq M$$

→ once that $d \in D$ has been fixed, we choose $R_0 \in \mathbb{N}$ such that (2) is $\leq \frac{1}{4} \varepsilon$ for every $k \geq R_0$, $R \geq R_0$ (possible because $\{f_{n_k}(d)\}$ is a Cauchy seq.)

Step 3 At the end of step 2 we know that

$$\lim_{k \rightarrow \infty} f_{n_k}(v) = \text{something} =: f_\infty(v) \quad \forall v \in V$$

We need that $f_\infty \in V'$. This is easy

→ f_∞ is linear (pointwise limit of linear functions)

→ f_∞ is continuous (pointwise limit of M -lip. cont. func.)

— o — o —

Exercise Try to repeat the argument in order to show that ball is $\vee$ one weakly compact and see what happens.

Remark (About LSC of the norm)

Both in V and in V' the norm is the supremum of linear functions. As a general fact, the sup of cont. functions is LSC.

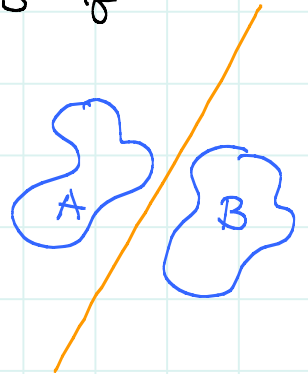
— o — o —

SEPARATION RESULTS

Def Let V be a normed space, and let A and B be two non-empty subsets.

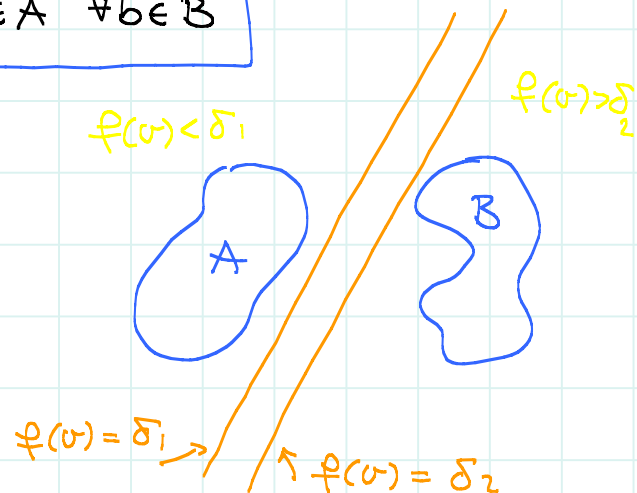
(1) We say that $f \in V'$ separates weakly A and B if

$$f(a) < f(b) \quad \forall a \in A \quad \forall b \in B$$



(2) We say that $f \in V'$ separates strongly A and B if there exists $\delta_1 < \delta_2$ such that

$$f(a) \leq \delta_1 < \delta_2 \leq f(b) \quad \forall a \in A \quad \forall b \in B$$



Theorem 1 (Weak separation of convex sets)

Let us assume that

(i) A and B are nonempty convex sets

(ii) $A \cap B = \emptyset$

(iii) A is OPEN

Then a weak separation exists

Theorem 2 (Strong separation of convex sets)

Let us assume that

(i) as before

(ii) as before

(iii) A is CLOSED and B is COMPACT

Then a strong separation exists.

Corollary (Strong + convex $\Rightarrow$ weak)

Let us assume that

(i) A is strongly closed

(ii) A is convex

Then A is weakly closed.

Proof Assume that $A \ni U_n \rightarrow U_\infty$

Claim: $U_\infty \in A$

Assume it is not the case.

By strong sep. with $B = \{U_\infty\}$ we obtain that

$$f(U_n) \leq \delta_1 < \delta_2 \leq f(U_\infty)$$

$$\downarrow$$
$$f(U_\infty)$$

Note that a weak separation would not be enough.

— o — o —

