

Prop Let V be a normed space. Let $v_0 \in V$ be any element. Then there exists $f: V \rightarrow \mathbb{R}$ linear such that

$$f(v_0) = \|v_0\|_V$$

and

$$|f(v)| \leq \|v\|_V \quad \forall v \in V.$$

Proof Consider the (pseudo)-norm $p(v) = \|v\|$. Consider $E = \text{Span}(v_0)$ (if $v_0 = 0$, then $f \equiv 0$) and consider $f: E \rightarrow \mathbb{R}$ defined as

$$f(\alpha v_0) := \alpha \|v_0\|$$

Let us check that

$$f(v) \leq p(v) \quad \text{for every } v \in E$$

namely if $v = \alpha v_0$

$$\alpha \|v_0\| \leq \|\alpha v_0\| \quad \forall \alpha \in \mathbb{R}$$

which is trivial.

HB $\Rightarrow$ we can extend f to V in such a way that

$$f(v) \leq p(v) = \|v\|_V \quad \forall v \in V$$

We need that

$$|f(v)| \leq p(v) = \|v\|_V \quad \forall v \in V$$

If $f(v) \geq 0$ this is trivial

If $f(v) < 0$ then

$$|f(v)| = -f(v) = f(-v) \leq p(-v) = \|-v\|_V = \|v\|_V$$

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Def. In the sequel we call any such f **an** ALIGNED FUNCTIONAL of v_0 .

Answer (2) If V and W are normed vector spaces, then there exists a nontrivial $f: V \rightarrow W$ that is linear and continuous

Proof

$V \xrightarrow{\quad} \mathbb{R} \xrightarrow{\quad} W$
↑ ↑
aligned just map 1 to any $w_0 \neq 0$ in W
functional
of any $v_0 \neq 0$

SPACE OF CONTINUOUS LINEAR OPERATORS

Defn Let V and W be normed spaces.
We call $\mathcal{L}(V, W)$ the space of linear and continuous functions $f: V \rightarrow W$.
For every $f \in \mathcal{L}(V, W)$ we set

$$\|f\|_{\mathcal{L}(V, W)} := \sup \{ \|f(v)\|_W : v \in V \text{ and } \|v\|_V \leq 1 \}$$

Prop. Notations as above. Then $\mathcal{L}(V, W)$ is a vector space and $\|f\|_{\mathcal{L}(V, W)}$ is a norm in $\mathcal{L}(V, W)$, such that

$$\|f(v)\|_W \leq \|f\|_{\mathcal{L}(V, W)} \cdot \|v\|_V \quad \forall v \in V$$

Moreover, if W is complete (Banach), then also $\mathcal{L}(V, W)$ is complete (Banach).

Proof

Standard: vector space + norm + inequality

Let us check the completeness when W is complete.

Let $\{f_n\} \subseteq \mathcal{L}(V, W)$ be any Cauchy sequence, w.r.t the operator norm

Step 1 For every fixed $v \in V$ we have that $\{f_n(v)\}$ is a Cauchy sequence in W

$$\begin{aligned}\|f_n(v) - f_m(v)\|_W &= \|(f_n - f_m)(v)\|_W \\ &\leq \|f_n - f_m\|_{\mathcal{L}(V, W)} \cdot \|v\|_V \\ &\leq \varepsilon \text{ if } n \text{ and } m \text{ are large}\end{aligned}$$

Step 2 From step 1 we deduce that there exists $f_\infty: V \rightarrow W$ such that

$$f_n(v) \rightarrow f_\infty(v) \quad \forall v \in V$$

It is easy to see that

- f is linear (pointwise limit of lin. functions)
- f is continuous (pointwise limit of equi-lip. functions is Lip. cont. The seq. $\{f_n\}$ is equi-lip because Cauchy seq. are always bounded and the norm in $\mathcal{L}(V, W)$ is the Lip. constant).

Step 3 We have to prove that $f_n \rightarrow f_\infty$ wrt the norm of $\mathcal{L}(V, W)$.

$$\|(f_n - f_\infty)(v)\|_W \leq \varepsilon \quad \forall v \in V \text{ with } \|v\|_V \leq 1$$

if n is large enough.

Since it is a Cauchy sequence, we know that $\|f_n - f_m\|_{\mathcal{L}(V, W)} \leq \varepsilon$ for every $n \geq n_0, m \geq n_0$, namely

$$\|f_n(v) - f_m(v)\|_W \leq \varepsilon \quad \forall v \in V \text{ with } \|v\|_V \leq 1$$

Let $m \rightarrow \infty$.

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Proof by duality Let $L: \mathbb{R}^d \rightarrow \mathbb{R}$ be any linear map

Then

$$L \left(\int_a^b f(t) dt \right) = \int_a^b L(f(t)) dt$$

$\int$ commutes with L

and therefore

$$\left| L \left(\int_a^b f(t) dt \right) \right| = \left| \int_a^b L(f(t)) dt \right|$$

$$\leq \int_a^b |L(f(t))| dt$$

scalar function

$$\leq \int_a^b \|L\|_{V'} \cdot \|f(t)\|_V dt$$

$\uparrow$ dual $\uparrow \mathbb{R}^d$ with the given norm

$$\leq \int_a^b \|f(t)\|_V dt \quad \text{if } \|L\|_{V'} \leq 1$$

We take the supremum as L varies in V' with $\|L\|_{V'} \leq 1$
and in the LHS we obtain the norm.

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Example $V = L^1((0, +\infty))$

Define $L: V \rightarrow \mathbb{R}$ as

$$L(u) = \int_0^{+\infty} u(x) \arctan x \, dx$$

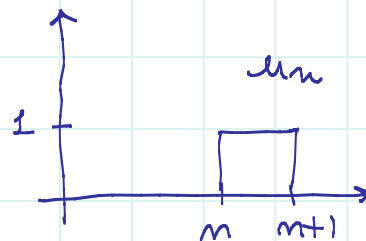
Can prove: L is linear and continuous because

$$|L(u)| \leq \frac{\pi}{2} \|u\|_V$$

therefore $L \in V'$ and $\|L\|_{V'} = \frac{\pi}{2}$

but sup is not realized.

In order to prove that $\|L\|_{V'} = \frac{\pi}{2}$ consider



Example 2 $V = L^1((0,2))$

$$L: V \rightarrow V$$

$$u(x) \rightarrow x u(x)$$

Then $\|L\|_{\mathcal{L}(V,V)} = 2$ and again it is not realized

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