

Fixed point theorems from BROUWER to SCHAUDER

Theorem (BROUWER) Let $R > 0$ and let $\bar{B}(0, R)$ be a closed ball in $\mathbb{R}^d$. Let

$$f: \bar{B}(0, R) \rightarrow \bar{B}(0, R)$$

be a continuous function.

Then there exists at least one $x \in \bar{B}(0, R)$ such that $f(x) = x$.

Proof Geometry courses (some algebraic topology is needed)

Corollary (Modified Brouwer theorem)

Let $D \subseteq \mathbb{R}^d$ be a CONVEX COMPACT set

Let $f: D \rightarrow D$ be a continuous functions.

Then f admits at least one fixed point.

Proof Take $R > 0$ s.t. $D \subseteq \bar{B}(0, R)$

Consider $P_D: \mathbb{R}^d \rightarrow D$ be the projection onto the convex set D .

Define

$$\hat{f}: \bar{B}(0, R) \rightarrow \bar{B}(0, R)$$

as

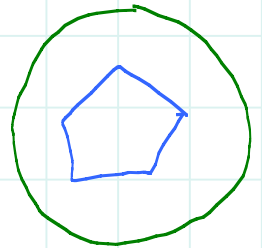
$$\hat{f}(x) := f(P_D(x))$$

Apply Brouwer to $\hat{f}$ and obtain $x \in \bar{B}(0, R)$ such that

$$x = \hat{f}(x) \in D$$

and therefore also $x \in D$.

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Bad news: there is no Brouwer theorem in infinite dimension

Example Let H be the usual separable Hilbert space with Hilbert basis $\{e_n\}$.

Let $(x_1, x_2, \dots)$ denote the components of $x \in H$.

Let us define

$$\varphi(x) = ((1 - \|x\|)^{1/2}, x_1, x_2, x_3, \dots) \quad \forall x \in \bar{B}(0, 1)$$

It is possible to check that φ is continuous, and that $\varphi: \bar{B}(0, 1) \rightarrow \bar{B}(0, 1)$ and actually $\|\varphi(x)\| = 1$ for every $x \in \bar{B}(0, 1)$.

On the other hand, φ has no fixed point

Any fixed point has to satisfy $x_1 = x_2 = x_3 = \dots$ and therefore $x = \underset{\substack{\uparrow \\ \text{unique vector with all equal components}}}{0}$, but $\varphi(0) \neq 0$

SCHAUDER FIXED POINT THEOREM

Setting:

$$\begin{array}{ccccc} Y & \supseteq & C & \supseteq & K \\ \uparrow & & \uparrow & & \uparrow \\ \text{normed} & & \text{convex set} & & \text{compact set} \\ \text{space} & & & & \end{array}$$

$\varphi: C \rightarrow K$ continuous function

Then φ admits at least one fixed point (of course in K).

Idea: approximate φ with function φ_n defined in space with finite dimension, when we can apply Brouwer

Proof] Apply the nonlinear projection lemma with $\varepsilon := \frac{1}{n}$.

We obtain

$$P_n : K \rightarrow K_n$$

↑
convex envelope of a finite
number of points of K

Observe that $K_n \subseteq C$ because C is convex, and therefore we can define

$$f_n : K_n \rightarrow K_n$$

as

$$f_n(x) = P_n(\underbrace{f(x)}_{\in K})$$

$\uparrow$
 $\in C$

We observe that f_n is continuous and K_n is contained in a vector space with finite dimension $Y_n = \text{Span}$ of the points whose convex envelope is K_n .

We observe that K_n is convex and compact, and from Brouwer we deduce that there exists $x_n \in K_n \subseteq C$ s.t.

$$f_n(x_n) = x_n.$$

Consider $\{f(x_n)\} \subseteq K$ and therefore $y_{n_k} \rightarrow y_\infty \in K$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad y_n$

Claim: y_∞ is a fixed point of f .

Before we prove that $x_{n_k} \rightarrow y_\infty$:

$$\|x_{n_k} - y_\infty\| \underset{\substack{\uparrow \\ \text{fixed point}}}{=} \|f_{n_k}(x_{n_k}) - y_\infty\|$$

$$\leq \|f_{n_k}(x_{n_k}) - f(x_{n_k})\| + \underbrace{\|f(x_{n_k}) - y_\infty\|}_{\substack{\uparrow y_{n_k} \\ \rightarrow 0}}$$

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$$\underbrace{\|P_{n_k}(f(x_{n_k})) - f(x_{n_k})\|}_{\leq \frac{1}{n_k} \text{ because of the property of the projection}}$$

At this point

$$\begin{array}{ccccccc} f(y_\infty) & = & f\left(\lim_{k \rightarrow \infty} x_{n_k}\right) & = & \lim_{k \rightarrow \infty} f(x_{n_k}) & = & \lim_{k \rightarrow \infty} y_{n_k} = y_\infty. \\ & & \text{---} \circ \text{---} \circ \text{---} & & & & \end{array}$$

Classical application

PEANO's theorem for diff. equations

Statement

Let $d \geq 1$ be an integer

Let $t_0 \in \mathbb{R}$, $u_0 \in \mathbb{R}^d$

Let $\delta_0 > 0$ and $r_0 > 0$

Let $f: \underbrace{[t_0 - \delta_0, t_0 + \delta_0] \times \bar{B}(u_0, r_0)}_{\mathbb{R}} \rightarrow \mathbb{R}^d$ continuous

Consider the Cauchy problem

$$u'(t) = f(t, u(t))$$

$$u(t_0) = u_0$$

Let

$$M := \max \{ |f(t, u)| : (t, u) \in \mathbb{R} \}$$

and let

$$\delta_1 \leq \delta_0 \quad \delta_1 M \leq r_0$$

Then there exists at least one solution

$$u: [t_0 - \delta_1, t_0 + \delta_1] \rightarrow \bar{B}(u_0, r_0)$$

Several proofs

- approximate f by Lipschitz functions
 - delay problems "à la Tonelli"
 - Schauder
- } see lectures "Analisi 2"

As usual, a solution is a fixed point of the VOLTERRA operator

$$[\Phi(u)](t) := u_0 + \int_{t_0}^t f(s, u(s)) ds \quad \forall t \in [t_0 - \delta_1, t_0 + \delta_1]$$

Functional setting

$$\mathcal{Y} := \{ \text{continuous functions } u: [t_0 - \delta_1, t_0 + \delta_1] \rightarrow \mathbb{R}^d \}$$

↑ normed vector space (sup norm)

$$\mathcal{C} := \{ \text{cont. functions } u: [t_0 - \delta_1, t_0 + \delta_1] \rightarrow \bar{B}(u_0, r_0) \}$$

↑ convex set

$$\mathcal{K} := \{ \text{functions in } \mathcal{C} \text{ that are Lip. cont. with constant } \leq M \}$$

↑ compact set because of Ascoli-Arzelà

We just need to prove that $\Phi: \mathcal{C} \rightarrow \mathcal{K}$

(here we need the definition of M and condition $\delta_1 M \leq r_0$)

We need also that Φ is continuous wrt the norm of $\mathcal{Y}$

(true because integrals pass to the limit wrt uniform convergence).

Schauder $\rightarrow$ at least one fixed point.



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