

APPROXIMATION OF COMPACT OPERATORS

Prop. Setting: X and Y are metric spaces

$$f_n: X \rightarrow Y$$

$$f_\infty: X \rightarrow Y$$

Assumptions:

(i) Y IS COMPLETE $\leftarrow$ [Exercise: show that this is needed]

(ii) f_n is a compact operator for every $n \geq 1$

(iii) $f_n \rightarrow f_\infty$ unif. on bounded subsets of X .

Then f_∞ is compact.

Proof Let $B \subseteq X$ be any bounded set. We need to prove that $f_\infty(B)$ is relatively compact in Y .

Since Y is complete, it is enough to prove that $f_\infty(B)$ is totally bounded.

Idea: $f_\infty(B)$ is close to $f_n(B)$, which is totally bounded

Details.

• We are given $\varepsilon > 0$.

• There exists n large enough so that

$$d_Y(f_\infty(x), f_n(x)) \leq \frac{1}{2}\varepsilon \quad \forall x \in B$$

• There exist finitely many points $y_1, \dots, y_k$ such that

$$f_n(x) \in \bigcup_{i=1}^k B(y_i, \frac{1}{2}\varepsilon) \quad \forall x \in B$$

• By the triangular inequality $f_\infty(x) \in \bigcup_{i=1}^k B(y_i, \varepsilon) \quad \forall x \in B$.

Let H be a **SEPARABLE** Hilbert space, and let $\{e_n\}$ be a Hilbert basis.

Let

$$H_n := \text{Span}(e_1, \dots, e_n) \quad \leftarrow \text{vector space of dim. } n$$

Let

$$P_n : H \rightarrow H_n$$

denote the orthogonal projection.

If X is any metric space, and $f : X \rightarrow H$ is any function, we can set

$$f_n(x) = P_n(f(x)) \quad \forall x \in X$$

Then

$$f_n : X \rightarrow H_n \quad (\text{finitely dim. approx. of } f)$$

Question : in which sense $f_n \rightarrow f$?

Trivial answer : pointwise convergence, namely $f_n(x) \rightarrow f(x)$ for every $x \in X$.

Prop (Approximation of compact operators in Hilbert space)

Setting as above.

- (1) If f is compact, then $f_n \rightarrow f$ uniformly on bounded subsets of X
- (2) If f is continuous, then f_n is continuous (trivial)
- (2) If X is a normed space, and f is linear, then f_n is linear (trivial)

Proof. Let $B \subseteq X$ be a bounded subset, and let $\varepsilon > 0$.

By assumption $f(B) \subseteq H$ is rel. compact and therefore totally bounded, and therefore there exists $v_1, \dots, v_k$ in H such that

$$\varphi(B) \subseteq \bigcup_{i=1}^k B(u_i, \frac{1}{2}\varepsilon)$$

For every $i = 1, \dots, k$ there exists $z_i \in \text{Span}(e_1, \dots, e_n, \dots)$ such that

$$\|u_i - z_i\| \leq \frac{1}{2} \varepsilon$$

Finally, there exists $m_0 \geq 1$ such that $z_i \in H_{m_0}$ for every $i = 1, \dots, k$.

Claim: $\|f(x) - f_n(x)\| \leq \varepsilon \quad \forall n \geq m_0 \quad \forall x \in B$

Given $x \in B$, find i such that $f(x) \in B(v_i, \frac{1}{2}\epsilon)$ and observe that

$$\|f(x) - f_n(x)\| = \|f(x) - P_n(f(x))\|$$

$$P_m(f(x)) \min_{z_i \in H_m} \|f(x) - z_i\|$$

minimizes the distance from H_m

$$\leq \|f(x) - v_i\| + \|v_i - z_i\|$$

$$\leq \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon$$

[Remark] This result motivates some results "IF and ONLY IF" such as

" $f: X \rightarrow H$ is <assumption> if and only if f can be approximated by a sequence $f_n: X \rightarrow H_n$ satisfying <assumption> "

For example "Linear, continuous compact"

" f_n linear, continuous, compact"

compactness follows from boundedness

if the target space has finite dim

NONLINEAR PROJECTION

Theorem

Let Y be a normed space

Let $K \subseteq Y$ be a compact subset

Let $\varepsilon > 0$.

Then there exists $K_\varepsilon \subseteq Y$ and $P_\varepsilon : K \rightarrow K_\varepsilon$ such that

(1) K_ε is the convex envelope of a finite number of points of K

(2) P_ε is continuous

(3) $\|P_\varepsilon(y) - y\| \leq \varepsilon$ for every $y \in K$

[ε removed after video]

Proof

Since K is compact, there exist $y_1, \dots, y_k$ such that

$$K \subseteq \bigcup_{i=1}^k B(y_i, \varepsilon) = A_\varepsilon$$

Let K_ε be the convex envelope of these points

$$K_\varepsilon = \left\{ \sum_{i=1}^k \lambda_i y_i : \lambda_i \geq 0 \text{ and } \sum_{i=1}^k \lambda_i = 1 \right\}$$

For every $i = 1, \dots, k$ we introduce the function the "small cone function"



$$c_i(y) := \max \{ 0, \varepsilon - \|y - y_i\| \}$$

Let us consider $S(y) := \sum_{i=1}^k c_i(y) \neq 0$ for every $y \in A_\varepsilon$.

For every $y \in A_\varepsilon$ let us define

$$P_\varepsilon(y) := \underbrace{\sum_{i=1}^k \frac{c_i(y)}{S(y)} y_i}_{\in K_\varepsilon \text{ because sum of coeff.} = 1} \quad \forall y \in A_\varepsilon$$

We only need to check that $\|P_\varepsilon(y) - y\| \leq \varepsilon$ for every $y \in K$

$$\|P_\varepsilon(y) - y\| = \left\| \sum_{i=1}^k \frac{c_i(y)}{S(y)} y_i - y \right\|$$

$$= \left\| \sum_{i=1}^k \frac{c_i(y)}{S(y)} (y_i - y) \right\|$$

$$\leq \sum_{i=1}^k \frac{c_i(y)}{S(y)} \underbrace{\|y_i - y\|}_{\substack{\uparrow \text{this is less than } \varepsilon \\ \text{when } c_i(y) \neq 0}}$$

$$\leq \sum_{i=1}^k \frac{c_i(y)}{S(y)} \varepsilon$$

$$= \varepsilon.$$

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Remark In many statements P_ε or better $P_{1/n}$ plays the role of the projection $P_n: H \rightarrow H_n$ in Hilbert.

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