

SPECTRAL THEOREM (for compact operators)

Theorem let H be a separ. Hilbert space. let $f: H \rightarrow H$ and let us assume that ^{↑ (infinite dimension)}

- (i) f is linear
- (ii) f is compact
- (iii) f is symmetric

Then the following statements are true.

- (1) H admits a Hilbert basis $\{v_n\}$ made by eigenvectors of f
 $f(v_n) = \lambda_n v_n \quad \forall n \geq 1$
- (2) The sequence $\{\lambda_n\}$ is such that
 $|\lambda_n| \rightarrow 0$
- (3) For every $n \geq 1$, the eigenspace of λ_n has FINITE dimension unless $\lambda_n = 0$

Proof Step 1 Eigenvectors with different eigenvalues are orthogonal

$$f(v) = \lambda v \quad f(w) = \mu w$$

$$\begin{aligned} \lambda \langle v, w \rangle &= \langle \lambda v, w \rangle = \langle f(v), w \rangle = \langle v, f(w) \rangle \\ &= \langle v, \mu w \rangle = \mu \langle v, w \rangle \end{aligned}$$

$$\leadsto (\lambda - \mu) \langle v, w \rangle = 0$$

$\neq 0$

Step 2 If V is f -invariant, then $V^\perp$ is again f -invariant

$w \in V^\perp$ and $v \in V$. Then

$$\langle f(w), v \rangle = \langle w, \underbrace{f(v)}_{\in V} \rangle = 0 \quad \Rightarrow \quad f(w) \in V^\perp$$

Step 3 The idea is to consider a sequence of f -invariant subspaces

$$H = V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$$

and in each subspace we apply the variational characterization.

$$V_1 = H \leadsto \text{there exists } u_1 \text{ and } \lambda_1 \text{ s.t. } f(u_1) = \lambda_1 u_1$$

$$\text{Set } V_2 = \{v \in H : \langle v, u_1 \rangle = 0\} \text{ . This is } f\text{-invariant} \\ (\perp \text{ of } \text{Span}(u_1))$$

$$\leadsto \text{there exists } u_2 \text{ and } \lambda_2 \text{ s.t. } f(u_2) = \lambda_2 u_2$$

$$\text{Note that } |\lambda_2| \leq |\lambda_1| \\ \uparrow \quad \quad \uparrow \text{sup} \{ |q(u)| : u \in V_1 \setminus \{0\} \} \\ \text{sup} \{ |q(u)| : u \in V_2 \setminus \{0\} \}$$

... Suppose we have defined $u_1, \dots, u_k$. Then set

$$V_{k+1} = \{v \in V : \langle v, u_i \rangle = 0 \text{ for every } i = 1, \dots, k\} \\ = \text{Span}(u_1, \dots, u_k)^\perp \text{ and therefore } f\text{-invariant}$$

Actually there are two cases.

$\rightarrow$ there exists $k \geq 1$ such that $q(u) \equiv 0$ in V_k

$\rightarrow q(u) \not\equiv 0$ for every $k \geq 1$.

Case 1 $q(u) \equiv 0$ in V_k . Then $V_k = \text{Ker}(f)$

$$\text{Hyp. } \langle f(v), v \rangle = 0 \quad \forall v \in V$$

$$\text{Th } f(v) = 0 \quad \forall v \in V$$

Here we used symmetry (think of rotation by 90° degrees in $\mathbb{R}^2$)

$$\begin{aligned} 0 &= \langle f(v+w), v+w \rangle \\ &= \langle f(v), v \rangle + 2 \langle f(v), w \rangle + \langle f(w), w \rangle = 2 \langle f(v), w \rangle \end{aligned}$$

$\uparrow$ symmetry $\overset{0}{\parallel}$ $\overset{0}{\parallel}$

for every v and w in $V \leadsto f(v) = 0$
because $f(v) \in V \cap V^\perp$

If we are in this case then we take $v_1, \dots, v_{k-1}$ and we add an orthonormal basis of $\ker(f)$

$\uparrow$ this is to turn a separable Hilbert space

(general fact: any subset of a separ. metric space is separ.)

Case 2 We obtain an infinite sequence $\{v_n\}$ s.t.

$$f(v_n) = \lambda_n v_n$$

This is NOT YET an orthonormal basis.

We observe that $|\lambda_{n+1}| \leq |\lambda_n|$ for every $n \geq 1$

The claim is that $\lambda_n \rightarrow 0$.

If this is not the case, then $|\lambda_n| \geq \gamma_0 > 0$

Consider $\{v_n\}$. It is a bounded sequence. Therefore

$\{f(v_n)\} = \{\lambda_n v_n\}$ has a converging subseq.

On the other hand

$$\begin{aligned} \|\lambda_n v_n - \lambda_m v_m\|^2 &= \lambda_n^2 \underbrace{\|v_n\|^2}_1 + \lambda_m^2 \underbrace{\|v_m\|^2}_1 + 2 \lambda_n \lambda_m \underbrace{\langle v_n, v_m \rangle}_0 \\ &\geq 2\gamma_0^2 \end{aligned}$$

$\Rightarrow \{\lambda_n v_n\}$ does NOT admit Cauchy subsequences

Step 4 Assume that $\lambda_n \rightarrow 0$ and consider

$$V_\infty := V_1 \cap V_2 \cap V_3 \cap \dots$$

$$= \text{Clos}(\text{Span}(v_1, \dots, v_k, \dots))^\perp$$

The claim is that V_∞ is $\text{Ker}(\varphi)$ and therefore it is enough to add a Hilbert basis of $\text{Ker}(\varphi)$.

Clearly $V_\infty \subseteq V_k$ for every $k \geq 1$, and therefore

$$\sup\{|q(v)| : v \in V_\infty \setminus \{0\}\} \leq \sup\{|q(v)| : v \in V_k \setminus \{0\}\} \\ = |\lambda_k| \rightarrow 0$$

$$\leadsto q(v) \equiv 0 \text{ in } V_\infty \leadsto (\text{as before}) \quad \varphi(v) \equiv 0 \text{ in } V_\infty.$$

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Example Let $\{e_n\}$ be a Hilbert basis in H
Define

$$\varphi(e_n) = \begin{cases} \frac{1}{n} e_n & \text{if } n \text{ is ODD} \\ 0 & \text{if } n \text{ is EVEN} \end{cases}$$

It can be verified that, if we extend φ by linearity, the resulting operator is compact with an infinite dimensional kernel.

Example (Compactness is needed) $H = L^2((0,1))$

Consider $A: H \rightarrow H$ defined by

$$[Au](x) = x u(x)$$

Fact 1 It is well-defined, linear and Lip. cont.

$$\begin{aligned}\|xu(x) - xv(x)\|_{L^2}^2 &= \int_0^1 x^2 (u(x) - v(x))^2 dx \\ &\leq \int_0^1 (u(x) - v(x))^2 dx \\ &= \|u(x) - v(x)\|_{L^2}^2\end{aligned}$$

Fact 2 It is symmetric

$$\langle Au, v \rangle = \int_0^1 x u(x) v(x) dx = \int_0^1 u(x) \cdot x v(x) dx = \langle u, Av \rangle$$

Fact 3 A admits no eigenvalue

$$\begin{aligned}\text{Assume } Au = \lambda u &\leadsto xu(x) = \lambda u(x) \text{ for a.e. } x \in (0,1) \\ u(x)(x - \lambda) &= 0 \\ &\leadsto u(x) = 0 \text{ almost everywhere}\end{aligned}$$

Fact 4 A is not compact (if it were, then eigenvalues would exist)

Exercise Find $\{u_n\} \subseteq L^2((0,1))$ bounded s.t. $\|u_n\| = 1$ but $\{xu_n(x)\}$ has no subseq. that converges STRONGLY. (No Cauchy subseq.)

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