

- Compact operators
- Symmetric operators
- SS and WS continuity
- RAYLEIGH quotient

Def. Let X and Y be metric spaces. A function $f: X \rightarrow Y$ is called **COMPACT** if the image of every bounded set in X is a relatively compact subset of Y .

In terms of sequences: if $\{x_n\} \subseteq X$ is bounded, then $\{f(x_n)\}$ admits a converging subsequence.

Remark Compact $\not\Rightarrow$ continuous [consider any function whose image is $\neq$ points].

Prop. (Linear + compact $\Rightarrow$ Strong - Strong continuous)

Let X and Y be normed vector spaces. Let $f: X \rightarrow Y$ be a function.

Let us assume that

(i) f is linear

(ii) f is compact.

Then f is SS continuous, and actually also lip. cont., and in particular

$$x_n \rightarrow x_\infty \text{ in } X \Rightarrow f(x_n) \rightarrow f(x_\infty) \text{ in } Y.$$

(strong convergence)

Proof. Since f is linear, we can assume wlog that $x_\infty = 0$.

Let $x_n \rightarrow 0$. We know that it is bounded.

Assume that $x_n \neq 0$ for every $n \in \mathbb{N}$.

Then consider $\frac{x_n}{\|x_n\|_X} = v_n$. Then $\|v_n\| = 1$ and therefore v_n is bounded

$\Rightarrow f(v_n)$ admits a converging subsequence

$\Rightarrow \|f(v_n)\|_Y \leq M$ for every $n \in \mathbb{N}$

$$\|f(v_n)\|_Y = \frac{\|f(x_n)\|_Y}{\|x_n\|_X} \leq M$$

$$\Rightarrow \|f(x_n)\|_Y \leq M \|x_n\|_X \quad (\text{true for free even if } x_n = 0)$$

$\downarrow$
0

$$\Rightarrow \|f(x_n)\|_Y \rightarrow 0$$

This proves that every subsequence $f(x_{n_k})$ has a further subsequence s.t. $f(x_{n_{k_i}}) \rightarrow 0$.

By the sub-sub lemma this implies that $f(x_n) \rightarrow 0$.

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Def. (Symmetric operator)

Let H be a Hilbert space. A function $f: H \rightarrow H$ is called symmetric if

$$\langle f(v), w \rangle = \langle v, f(w) \rangle \quad \forall (v, w) \in H^2$$

Exercise If f is symmetric, then f is LINEAR

Hint

$$\begin{aligned} \langle f(v_1 + v_2), w \rangle &= \langle v_1 + v_2, f(w) \rangle \\ &= \langle v_1, f(w) \rangle + \langle v_2, f(w) \rangle \\ &= \langle f(v_1), w \rangle + \langle f(v_2), w \rangle \\ &\dots \end{aligned}$$

Prop. (Lin. + symm. + cpt. $\Rightarrow$ weak-strong continuous)

Let H be a Hilbert space and let $f: H \rightarrow H$ be a function.

Let us assume that

(i) f is linear

(ii) f is compact

(iii) f is symmetric

Then f is WS continuous, namely (if x_n is bounded and)

$$\underset{\text{weak}}{x_n \rightarrow x_\infty \text{ in } H} \Rightarrow \underset{\text{strong}}{f(x_n) \rightarrow f(x_\infty) \text{ in } H}$$

Proof

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f(x_n), w \rangle & \stackrel{f \text{ symm}}{=} \lim_{n \rightarrow \infty} \langle x_n, f(w) \rangle \\ & \stackrel{x_n \rightarrow x_\infty}{=} \langle x_\infty, f(w) \rangle \\ & \stackrel{f \text{ symm}}{=} \langle f(x_\infty), w \rangle \end{aligned}$$

This proves that $f(x_n) \rightarrow f(x_\infty)$ weakly

Take any subsequence $\{f(x_{n_k})\}$. Due to compactness, this admits a further subsequence

$$f(x_{n_{k_i}}) \rightarrow y_\infty \text{ strongly}$$

But we know that

$$f(x_{n_{k_i}}) \rightarrow f(x_\infty)$$

and therefore $y_\infty = f(x_\infty)$.

The conclusion follows from the sub-sub lemma

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Remark One day the same statement is true without $\{x_n\}$ bounded and even without assuming f is symmetric.

RAYLEIGH QUOTIENT Let $\varphi: H \rightarrow H$

$$q(v) := \frac{\langle \varphi(v), v \rangle}{\|v\|^2} \quad \forall v \in H \setminus \{0\}$$

Theorem (Variational characterization of eigenvalues/eigenvectors)

Let $\varphi: H \rightarrow H$ as above. Let us assume that

(i) φ is linear

(ii) φ is symmetric

(iii) φ is compact

strongly
↑

Let $V \subseteq H$ be a φ -invariant closed subspace ($\varphi(v) \in V$ for every $v \in V$).

Then the following are true

(1) There exists

$$\max \{ |q(v)| : v \in V \setminus \{0\} \}$$

and it coincides with

$$\sup \{ |\langle \varphi(v), v \rangle| : \|v\| = 1, v \in V \}$$

and with

$$\sup \{ |\langle \varphi(v), v \rangle| : \|v\| \leq 1, v \in V \}$$

(2) Consider any maximum point $v_0 \in V \setminus \{0\}$ and let $|q(v_0)|$ be the maximum value.

Then

$$\varphi(v_0) = q(v_0) \cdot v_0$$

↑ *eigenvalue* ← *eigenvector*

without absolute value

Remark We do not say that $q(v)$ admits max and min. This is true only in **FINITE** dimension

Exercise Find example

Proof (1) We claim that

$$\sup \{ |q(v)| : v \in V \setminus \{0\} \} = \sup \{ |\langle f(v), v \rangle| : v \in V, \|v\| = 1 \}$$

$\uparrow$
 $q(v)$ is $\mathcal{O}$ -homogeneous

$$\leq \sup \{ |\langle f(v), v \rangle| : v \in V, \|v\| \leq 1 \}$$

The key point is that the last one is an equality, because for every $w \in V$ with $\|w\| < 1$ we find that $\frac{w}{\|w\|}$ is a better competitor because

$$\left| \left\langle f\left(\frac{w}{\|w\|}\right), \frac{w}{\|w\|} \right\rangle \right| = \frac{1}{\|w\|^2} \left| \langle f(w), w \rangle \right| \geq \left| \langle f(w), w \rangle \right|$$

$\uparrow$ lin. $\uparrow$ $\|w\|^2 < 1$

[Some discussion needed if $w = 0$]

The second important point is that the last one is a maximum, namely

$$\max \{ |\langle f(v), v \rangle| : v \in V, \|v\| \leq 1 \} \text{ exists}$$

Take $\{v_n\}$ a sup-izing sequence. Then there exists

$$v_{n_k} \rightarrow v_\infty \text{ weakly}$$

By the ws continuity we know that

$$f(v_{n_k}) \rightarrow f(v_\infty) \text{ strongly}$$

$$\langle f(v_{n_k}), v_{n_k} \rangle \rightarrow \langle f(v_\infty), v_\infty \rangle$$

$\uparrow$
 $\downarrow$ weak
 $\downarrow$ strong

(2) Let v_0 be a maximum point for $|q(v)|$.

Then v_0 is either max or min for $q(v)$

Write

$$q(v_0 + tv) = \frac{\langle f(v_0 + tv), v_0 + tv \rangle}{\|v_0 + tv\|^2}$$
$$= \frac{\langle f(v_0), v_0 \rangle + 2t \langle f(v_0), v \rangle + t^2 \langle f(v), v \rangle}{\|v_0\|^2 + 2t \langle v_0, v \rangle + t^2 \|v\|^2}$$

$$0 = q'(0) = \frac{2 \langle f(v_0), v \rangle \|v_0\|^2 - 2 \langle v_0, v \rangle \langle f(v_0), v_0 \rangle}{\|v_0\|^4}$$

$\forall v \in V$

$$\underbrace{\langle f(v_0) \|v_0\|^2 - \langle f(v_0), v_0 \rangle v_0, v \rangle}_{=0} = 0 \quad \forall v \in V$$

$$f(v_0) = \frac{\langle f(v_0), v_0 \rangle}{\|v_0\|^2} v_0 = q(v_0) v_0$$

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Questions $\rightarrow$ Where did we use that we have $|q(v)|$?

$\rightarrow$ Where did we use that V is f -invariant?