

Boundary regularity for Dirichlet problem

Theorem

Let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be a solution to $\Delta u = f$ in the usual weak sense.

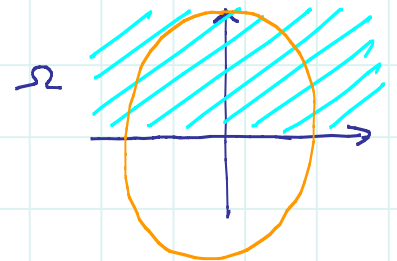
Assume that $\partial\Omega$ is compact and of class C^2 .

Then $u \in H^2(\Omega)$ and

$$\|D^2 u\|_{L^2(\Omega)} \leq c(\Omega) \{ \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \}$$

The same is true for $\operatorname{div}(A(x)\nabla u) = f$ provided that $A \in C^1(\operatorname{clos}(\Omega))$ and unif. elliptic.

Model case $\Omega = \mathbb{R}_+^d$



Proof Let us show that a priori estimates

$$\begin{matrix} (x, y) \\ \mathbb{R}^{d-1} \uparrow y \in \mathbb{R}_+ \end{matrix}$$

Let us assume that f and u are in $C^\infty(\mathbb{R}_+^d)$ and they vanish outside a large enough ball of $\mathbb{R}^d$.

For every $i = 1, \dots, d-1$ we use $\varphi = u x_i x_i$

$$- \int_{\mathbb{R}_+^d} \langle \nabla u, \nabla u x_i x_i \rangle dx = \int_{\mathbb{R}_+^d} f u x_i x_i dx$$

↑
we can integrate by parts w.r.t x_i because we do not cross the boundary

As in the case of $\mathbb{R}^d$ we obtain that

$$\sum_{j=1}^d \|u_{x_i x_j}\|_{L^2(\mathbb{R}_+^d)}^2 \leq \|f\|_{L^2(\mathbb{R}_+^d)}^2 \quad i=1, \dots, d-1$$

In this way we can estimate all second order derivatives of u with the exception of u_{yy} . But

$$u_{yy} = f - \sum_{i=1}^{d-1} u_{x_i x_i} \Rightarrow u_{yy} \in L^2 \text{ with estimate}$$

In the case where $f \in L^2$ and $u \in H^1$ we use discrete derivatives in the same way as for the internal regularity.

$$\varphi = D_{x_i}^{-\varepsilon} D_{x_i}^{\varepsilon} u \in H_0^1(\mathbb{R}_+^d) \text{ because } i=1, \dots, d-1$$

Again we obtain that all second der. of u , but u_{yy} , are in $L^2(\mathbb{R}_+^d)$.

$$\begin{aligned} - \int_{\Omega} f \varphi &= + \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \\ &= \sum_{i=1}^{d-1} \int_{\Omega} D_{x_i} u D_{x_i} \varphi + \int_{\Omega} D_y u D_y \varphi \\ &\quad \downarrow \text{here we can integrate by parts} \end{aligned}$$

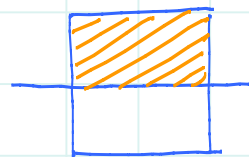
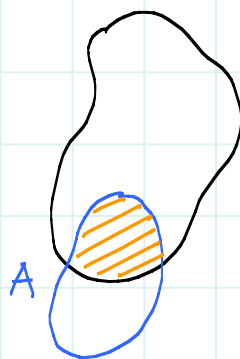
$$= - \sum_{i=1}^{d-1} \int_{\Omega} D_{x_i x_i} u \cdot \varphi + \int_{\Omega} D_y u D_y \varphi$$

$$- \int_{\Omega} D_y u D_y \varphi = \int_{\Omega} \underbrace{\{ f - D_{x_i x_i} u \}}_{g \in L^2} \varphi$$

This is equivalent to saying that $D_y u$ has a weak deriv wrt y and $D_y D_y u = g$

General case

As always we write u as a sum of functions u_i with support in suitable open sets that are diffeomorphic to the standard cube.



Problem Consider two diffeom. open sets A and B

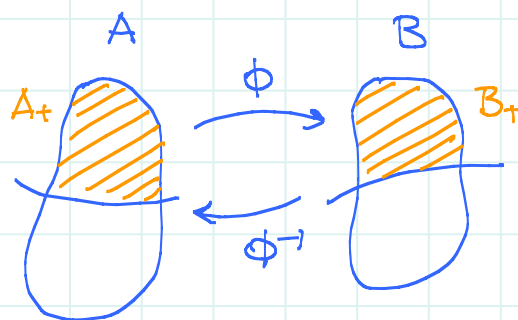
$$\phi: A \rightarrow B$$

$$\phi: A^+ \rightarrow B^+$$

$$\phi: \partial A^+ \rightarrow \partial B^+$$

If u solves

$$\operatorname{div}(\hat{A}(x) \nabla u) = f \quad \text{in } A_+$$



Let us define $v(y)$, $g(y)$, $\hat{B}(y)$ for $y \in B$ or B^+ such that

$$v(\phi(x)) = u(x)$$

$$g(\phi(x)) |\operatorname{Det} J\phi(x)| = f(x)$$

$$\hat{B}(\phi(x)) |\operatorname{Det} J\phi(x)| = J\phi(x) \hat{A}(x) J\phi(x)^t$$

We claim that $\operatorname{div}(\hat{B}(y) \nabla v(y)) = g(y)$ in B_+ in the usual weak sense

$$\underline{\text{Hyp}}: - \int_{A^+} \langle \hat{A}(x) \nabla u(x), \nabla \varphi(x) \rangle = \int_{A^+} f \varphi \quad \forall \varphi \in C_c^\infty(A^+)$$

$$\underline{\text{Th}}: - \int_{B^+} \langle \hat{B}(y) \nabla v(y), \nabla \psi(y) \rangle = \int_{B^+} g \psi \quad \forall \psi \in C_c^\infty(B^+)$$

$$\begin{aligned} \text{RHS} &= \int_{B_+} g(y) \psi(y) dy \stackrel{y=\phi(x)}{=} \int_{A_+} \underbrace{g(\phi(x))}_{\psi(\phi(x))} \underbrace{\psi(\phi(x))}_{\varphi(x)} \underbrace{|\text{Det } J\phi(x)| dx} \\ &= \int_{A_+} f(x) \varphi(x) dx \end{aligned}$$

Let us observe that $\nabla \varphi(x) = \nabla \psi(\phi(x)) J\phi(x)$

$$\boxed{\quad \quad \quad \square}$$

$$\nabla u(x) = \nabla v(\phi(x)) \cdot J\phi(x)$$

$$- \text{LHS} = \int_{B_+} \langle \hat{B}(y) \nabla v(y), \nabla \psi(y) \rangle dy$$

$$= \int_{B_+} \nabla v(y) \hat{B}(y) \nabla \psi(y)^t dy$$

$$= \int_{A_+} \nabla v(\phi(x)) \hat{B}(\phi(x)) \nabla \psi(\phi(x))^t |\text{Det } J\phi(x)| dx$$

$$= \int_{A_+} \underbrace{\nabla v(\phi(x)) J\phi(x)}_{\nabla u(x)} \hat{A}(x) \underbrace{J\phi(x)^t \nabla \psi(\phi(x))^t}_{\nabla \varphi(x)^t} dx$$

$$= - \text{LHS} \text{ in the assumption}$$

what we need is that

$$\hat{B}(\phi(x)) |\text{Det } J\phi(x)| = J\phi(x) \hat{A}(x) J\phi(x)^t$$

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Remark Where we need that $\partial\Omega$ is of class C^2 .
In at least two points

1 - We need $\hat{B}$ to be of class C^1

2 - We need that $v \in H^2 \Rightarrow u \in H^2$ and this requires ϕ to be of class C^2 .

Remark We need $\hat{B}$ to be unif. elliptic.

Lemma (lin. alg.) If A is coercive and M is invertible, then $MA M^t$ is coercive

Remark What about the case where $u = u_0$ on $\partial\Omega$ in the sense that $u - u_0 \in H_0^1$

$$\Delta(u - u_0) = \Delta u - \Delta u_0 = f - \Delta u_0$$

$$\begin{aligned} \text{If } u_0 \in H^2 \rightsquigarrow f - \Delta u_0 \in L^2 &\rightsquigarrow u - u_0 \in H^2 \\ &\rightsquigarrow u \in H^2. \end{aligned}$$

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