

DIFFERENCE QUOTIENTS

DISCRETE DERIVATIVES

Let $u: \mathbb{R}^d \rightarrow \mathbb{R}$, let $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $\varepsilon > 0$. We set

$$D_\alpha^\varepsilon u(x) := \frac{u(x + \varepsilon \alpha) - u(x)}{\varepsilon \|\alpha\|}$$

Diff. quotient
in direction α

If $\alpha = (0, \dots, 0, \underset{\substack{\uparrow \\ \text{pos. } i}}{1}, 0, \dots, 0) = e_i$ then we write $D_{x_i}^\varepsilon u(x)$

If u is defined in $\Omega \subseteq \mathbb{R}^d$ open set, then $D_\alpha^\varepsilon u$ is defined in every $\Omega' \subset\subset \Omega$ provide that ε is small enough.

SIMPLE PROPERTIES

(i) $u \rightarrow D_\alpha^\varepsilon u$ is linear

(ii) D_α^ε preserves regularity: if $u \in W^{m,p}(\mathbb{R}^d)$, then $D_\alpha^\varepsilon u \in W^{m,p}(\mathbb{R}^d)$

(iii) D_α^ε commutes with weak derivatives $D_\alpha^\varepsilon (D^\beta u) = D^\beta (D_\alpha^\varepsilon u)$
for every multiindex β .

(iv) There is a formula for the product

$$D_\alpha^\varepsilon (u(x)v(x)) = D_\alpha^\varepsilon u(x) \cdot v(x) + u(x + \varepsilon \alpha) \cdot D_\alpha^\varepsilon v(x)$$

(v) There is a formula for the composition. Assume $g \in C^1$

$$D_\alpha^\varepsilon (g(u(x))) = g'(\quad) D_\alpha^\varepsilon u(x)$$

↑ some point between $u(x)$ and $u(x + \varepsilon \alpha)$

Proposition

(1) If $u \in W^{1,p}(\mathbb{R}^d)$, then $\|D_{x_i}^{\varepsilon} u(x)\|_{L^p(\mathbb{R}^d)} \leq \|D_{x_i} u(x)\|_{L^p(\mathbb{R}^d)}$

(2) Discrete integration by parts

$$\int_{\mathbb{R}^d} D_\alpha^\varepsilon u(x) \cdot \varphi(x) dx = - \int_{\mathbb{R}^d} u(x) \cdot D_\alpha^{-\varepsilon} \varphi(x) dx$$

for every choice of u and φ for which the integrals make sense
for example

for example
 $\rightarrow u \in L^1_{loc}(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\rightarrow u \in L^p(\mathbb{R}^d) \text{ and } \varphi \in L^{p'}(\mathbb{R}^d) \text{ if } \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof (1) $|\mu(x + \varepsilon e_i) - \mu(x)| = |\varphi(1) - \varphi(0)|$
 $\uparrow$
 $\varphi(t) := \mu(x + \varepsilon e_i t)$

$$\leq \int_0^1 |\varphi'(t)| dt = \int_0^1 |D_{x_i} u(x + \varepsilon e_i t)| \cdot |\varepsilon| dt$$

$$\leq |\varepsilon| \left\{ \int_0^1 |D_{x_i} u(x + \varepsilon e_i t)|^p dt \right\}^{1/p}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x + \varepsilon e_i) - u(x)|^p dx &\leq |\varepsilon|^p \int_{\mathbb{R}^d} dx \int_0^1 |Dx_i u(x + \varepsilon e_i t)|^p dt \\ &= |\varepsilon|^p \int_0^1 dt \underbrace{\int_{\mathbb{R}^d} |Dx_i(\underbrace{x + \varepsilon e_i t}_y)|^p}_{dx = dy} dx \end{aligned}$$

$$= |\varepsilon|^p \int_{\mathbb{R}^d} |Dx_i(y)|^p dy$$

We divide by $|\varepsilon|^p$ and we get the conclusion

(2)

$$\int_{\mathbb{R}^d} \frac{u(x+\varepsilon d) - u(x)}{\varepsilon \|d\|} \varphi(x) dx$$

$$= \frac{1}{\varepsilon \|d\|} \left\{ \int_{\mathbb{R}^d} \underbrace{u(x+\varepsilon d)}_y \underbrace{\varphi(x)}_{\varphi(y-\varepsilon d)} dx - \int_{\mathbb{R}^d} u(x) \varphi(x) dx \right\}$$

$$= \frac{1}{-\varepsilon \|d\|} \left\{ \int_{\mathbb{R}^d} u(y) \varphi(y-\varepsilon d) dy - \int_{\mathbb{R}^d} u(y) \varphi(y) dy \right\}$$

This proves (2).

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Theorem (Sobolev spaces and discrete derivatives)

Let us assume that $u \in L^p(\mathbb{R}^d)$ with $p \in (1, +\infty)$

↑ excluded, but something is true

Then the following are equivalent

(1) $u \in W^{1,p}(\mathbb{R}^d)$

(2) $D_{x_i}^\varepsilon u \rightarrow \text{something in } L^p(\mathbb{R}^d) \text{ as } \varepsilon \rightarrow 0^+ \text{ for every } i=1, \dots, d$
↑ a posteriori is $D_{x_i} u$

(3) $D_{x_i}^\varepsilon u \rightarrow \text{something weakly in } L^p(\mathbb{R}^d)$

(4) $\|D_{x_i}^\varepsilon u\|_{L^p(\mathbb{R}^d)}$ are bounded indep. of ε

Proof (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are almost trivial by general facts

(1) $\Rightarrow$ (2) less easy, but again an exercise

[Hint: very similar to the proof that $u * \rho_\varepsilon \rightarrow u$]

(4) $\Rightarrow$ (1) Due to the bound on the norms, there exists $\varepsilon_k \rightarrow 0^+$ such that (we need also $p > 1$)

$$D_{x_i}^{\varepsilon_k} u(x) \longrightarrow v_i(x) \text{ weakly in } L^p(\mathbb{R}^d)$$

Claim: $v_i(x) = D_{x_i} u(x)$, namely

$$\int_{\mathbb{R}^d} u(x) D_{x_i} \varphi(x) dx = - \int_{\mathbb{R}^d} v_i(x) \varphi(x) dx$$

We start with the discrete integration by parts

$$\begin{aligned} \int_{\mathbb{R}^d} D_{x_i}^{\varepsilon_k} u(x) \varphi(x) dx &= - \int_{\mathbb{R}^d} u(x) D_{x_i}^{-\varepsilon_k} \varphi(x) dx \\ &\quad \downarrow \qquad \qquad \qquad \downarrow \text{also uniformly if } \varphi \in C_c^\infty(\mathbb{R}^d) \\ \int_{\mathbb{R}^d} v_i(x) \varphi(x) dx &= - \int_{\mathbb{R}^d} u(x) D_{x_i} \varphi(x) dx \end{aligned}$$

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Theorem (Internal regularity)

Assume that $f \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)$ is a solution to

$$\Delta u = f$$

in the weak formulation

$$- \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle dx = \int_{\Omega} f \varphi$$

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d)$$

$$\forall \varphi \in H^1(\mathbb{R}^d)$$

Then $u \in H^2(\mathbb{R}^d)$ and

$$\| D^2 u \|_{L^2(\mathbb{R}^d)}^2 \leq d \| f \|_{L^2(\mathbb{R}^d)}^2$$

Proof Last lecture we used $\varphi = dx_i \cdot x_i$.

Now we use

$$\varphi = D_{x_i}^{-\varepsilon} D_{x_i}^{\varepsilon} u(x) \in H^1(\mathbb{R}^d)$$

$$-\int_{\mathbb{R}^d} \langle \nabla u(x), \nabla D_{x_i}^{-\varepsilon} D_{x_i}^{\varepsilon} u(x) \rangle dx = \int_{\mathbb{R}^d} \varphi D_{x_i}^{-\varepsilon} D_{x_i}^{\varepsilon} u(x) dx$$

$$\begin{aligned} \text{RHS} &\leq \|\varphi\| \cdot \|D_{x_i}^{-\varepsilon} D_{x_i}^{\varepsilon} u\| \stackrel{\uparrow \text{Prop}}{\leq} \|\varphi\| \cdot \|D_{x_i} D_{x_i}^{\varepsilon} u\| \\ &\leq \|\varphi\| \cdot \|\nabla D_{x_i}^{\varepsilon} u\| \\ &= \|\varphi\| \cdot \|D_{x_i}^{\varepsilon} \nabla u\| \end{aligned}$$

$$\text{LHS} = - \int_{\mathbb{R}^d} \langle \nabla u(x), \nabla D_{x_i}^{-\varepsilon} D_{x_i}^{\varepsilon} u(x) \rangle dx$$

$$= - \int_{\mathbb{R}^d} \langle \nabla u(x), D_{x_i}^{-\varepsilon} \nabla D_{x_i}^{\varepsilon} u(x) \rangle dx$$

$$\begin{aligned} &\stackrel{\substack{\uparrow \\ \text{integr. by parts} \\ \text{on each term} \\ \text{of the sc. prod}}}{=} \int_{\mathbb{R}^d} \langle D_{x_i}^{\varepsilon} \nabla u(x), \nabla D_{x_i}^{\varepsilon} u(x) \rangle dx \\ &= \int_{\mathbb{R}^d} \langle D_{x_i}^{\varepsilon} \nabla u(x), D_{x_i}^{\varepsilon} \nabla u(x) \rangle dx \\ &= \|D_{x_i}^{\varepsilon} \nabla u(x)\|^2 \end{aligned}$$

$$\text{In conclusion} \quad \|D_{x_i}^{\varepsilon} \nabla u(x)\|^2 \leq \|\varphi\| \cdot \|\cancel{D_{x_i}^{\varepsilon} \nabla u(x)}\|$$

$$\text{This implies that} \quad \|D_{x_i} \nabla u(x)\|^2 \leq \|\varphi\|^2$$

$\uparrow \quad \sum_{j=1}^d \|D_{x_i} x_j u(x)\|^2$

Sum over i and get conclusion.