

Theorem (Internal regularity)

Let $d \geq 1$, let $\Omega \subseteq \mathbb{R}^d$ be any open set. Let us assume that

$$\operatorname{div}(A(x)\nabla u(x)) = f(x) \quad \text{in } \Omega$$

in the sense that

$$-\int_{\Omega} \langle A(x)\nabla u(x), \nabla \varphi(x) \rangle dx = \int_{\Omega} f(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

Let us assume that

- (i) $f \in L^2_{\text{loc}}(\Omega)$
 - (ii) $A(x) \in C^1(\Omega)$
 - (iii) $u \in H^1_{\text{loc}}(\Omega)$
 - (iv) $A(x)$ unip. elliptic in Ω
- under these assumptions
the weak formulation
makes sense

Then $u \in H^2_{\text{loc}}(\Omega)$ and for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$

$$\|D^2 u\|_{L^2(\Omega_1)} \leq c(\Omega_1, \Omega_2) \{ \|f\|_{L^2(\Omega_2)} + \|u\|_{L^2(\Omega_2)} \}$$

UN-NATURAL ESTIMATE



Remark The theorem is true also in iterated form,
namely

- (i) $f \in H^k_{\text{loc}}(\Omega)$
- (ii) $A \in C^{k+1}$
- (iii) $u \in H^{k+1}_{\text{loc}}(\Omega)$

$$\leadsto u \in H^{k+2}_{\text{loc}}(\Omega)$$

A PRIORI ESTIMATES

Assume everything is C^∞ and prove the estimates

Baby case 1 $d=2$, $\Omega = \mathbb{R}^2$, $u \in C_c^\infty(\mathbb{R}^2)$, $f \in C_c^\infty(\mathbb{R}^2)$

$$-\int_{\mathbb{R}^2} u_x \varphi_x + u_y \varphi_y = \int_{\mathbb{R}^2} f \varphi$$

Idea 1: use $\varphi = u_{xx}$ (I can because $u \in C_c^\infty$)

$$-\int_{\mathbb{R}^2} u_x u_{xxx} + u_y u_{xxy} = \int_{\mathbb{R}^2} f u_{xx}$$

$$\text{RHS} \leq \|f\|_{L^2} \cdot \|u_{xx}\|_{L^2} \leq \|f\|_{L^2} (\|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2)^{1/2}$$

$$\text{LHS} = \int_{\mathbb{R}^2} (u_{xx}^2 + u_{xy}^2) = \|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2$$

When I compare I get

$$\|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2 \leq \|f\|_{L^2} \cdot (\dots)^{1/2}$$

$$\text{and therefore } (\|u_{xx}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2)^{1/2} \leq \|f\|_{L^2}$$

If we use $\varphi = u_{yy}$ we obtain

$$(\|u_{yy}\|_{L^2}^2 + \|u_{xy}\|_{L^2}^2)^{1/2} \leq \|f\|_{L^2}$$

and finally

$$\|u_{xx}\|_{L^2}^2 + \|u_{yy}\|_{L^2}^2 + 2\|u_{xy}\|_{L^2}^2 \leq 2\|f\|_{L^2}^2$$

Baby case 2 any d , $\Omega = \mathbb{R}^d$, u and f in $C_c^\infty(\mathbb{R}^d)$

We use $\varphi = u_{x_i x_i}$ and in the same way we estimate all second derivatives with one x_i .

At the end of the day

$$\sum_{i=1}^d \|u_{x_i x_i}\|_{L^2}^2 + \sum_{i \neq j} \|u_{x_i x_j}\|_{L^2}^2 \leq d \|f\|_{L^2}^2$$

Baby case 3 any d , $\Omega = \mathbb{R}^d$, $u, f, A \in C_c^\infty(\mathbb{R}^d)$, equation

$$-\int_{\Omega} \langle A(x) \nabla u, \nabla \varphi \rangle = \int_{\Omega} f \varphi$$

Use again $\varphi = u_{x_i x_i}$

$$\begin{aligned} \text{RHS} &= \int_{\Omega} f u_{x_i x_i} \leq \|f\|_{L^2} \|u_{x_i x_i}\|_{L^2} \\ &\leq \|f\|_{L^2} \left\{ \sum_{j=1}^d \|u_{x_i x_j}\|_{L^2}^2 \right\}^{1/2} \end{aligned}$$

$$\text{LHS} = - \int_{\Omega} \langle A(x) \nabla u, \nabla u_{x_i x_i} \rangle dx$$

$$\begin{aligned} &= \int_{\Omega} \langle A_{x_i}(x) \nabla u, \nabla u_{x_i} \rangle dx + \underbrace{\int_{\Omega} \langle A(x) \nabla u_{x_i}, \nabla u_{x_i} \rangle}_{\geq 0 \text{ (ellipticity)}} \\ &\quad \geq \int_{\Omega} \|\nabla u_{x_i}\|^2 \end{aligned}$$

$$\int_{\Omega} \|\nabla u_{x_i}\|^2 \leq \|f\|_{L^2} \|\nabla u_{x_i}\|_{L^2}^2 - \int_{\Omega} \underbrace{\langle A_{x_i}(x) \nabla u, \nabla u_{x_i} \rangle}_{\text{bounded}}$$

In conclusion

$$\forall \|\nabla u_{x_i}\|_{L^2}^2 \leq \|f\|_{L^2} \|\nabla u_{x_i}\|_{L^2} + \|A\|_{C^1} \|\nabla u\|_{L^2} \|\nabla u_{x_i}\|$$

and we obtain

$$\sum_{j=1}^d \|\mu_{x_i x_j}\|_{L^2}^2 \leq \|f\|_{L^2} + \|A\|_{C^1} \cdot \|\nabla u\|_{L^2}$$

Case 4 $\Omega \subseteq \mathbb{R}^d$ any open set, $\Delta u = f$ in weak sense in Ω ,
 u and f in $C^\infty(\Omega)$ (no opt. support)

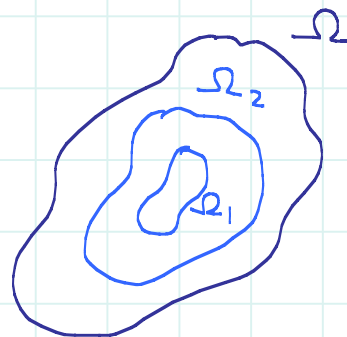
Consider $\Omega_1 \subset \subset \Omega_2 \subset \subset \Omega$

Consider $\psi \in C_c^\infty(\mathbb{R}^d)$ with

$\psi \equiv 1$ in Ω_1

$\psi \equiv 0$ outside Ω_2

$0 \leq \psi \leq 1$ in between



Define $v = u\psi$. Then

$$\Delta v = \Delta(u\psi) = \underbrace{\Delta u}_{=f} \psi + u \Delta \psi + 2 \langle \nabla u, \nabla \psi \rangle =: g$$

Then apply baby case 2 to v and g

$$\begin{aligned} \|D^2 u\|_{L^2(\Omega_1)}^2 &\stackrel{\substack{\uparrow \\ \mu = u \\ \text{in } \Omega_1}}{\leq} \|D^2 v\|_{L^2(\mathbb{R}^d)}^2 \leq d \|g\|_{L^2(\mathbb{R}^d)}^2 \\ &\stackrel{\substack{\uparrow \\ \text{depends on } \psi}}{\leq} \text{const} \left\{ \|f\|_{L^2(\Omega_2)}^2 + \|u\|_{L^2(\Omega_2)}^2 + \|\nabla u\|_{L^2(\Omega_2)}^2 \right\} \\ &\stackrel{\substack{\uparrow \\ \psi \equiv 0 \text{ outside } \Omega_2 \\ \text{and } \psi \leq 1 \text{ in } \Omega_2}}{\leq} \end{aligned}$$

Case 4 revisited Assume everything as in Case 4, but only $f \in L^2_{loc}$ and $u \in H^1_{loc}$

Can we say that v solves the same equation $\Delta v = g$?

Hyp:
$$-\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} f \varphi$$

Th:
$$-\int_{\Omega} \langle \nabla v, \nabla \varphi \rangle = \int_{\Omega} g \varphi$$

$$\begin{aligned} -\int_{\Omega} \langle \nabla v, \nabla \varphi \rangle &= -\int_{\Omega} \langle \nabla (u\varphi), \nabla \varphi \rangle \\ &= -\int_{\Omega} \langle \nabla u \cdot \varphi + u \nabla \varphi, \nabla \varphi \rangle \\ &= -\int_{\Omega} \langle \nabla u, \varphi \nabla \varphi \rangle - \int_{\Omega} \langle u \nabla \varphi, \nabla \varphi \rangle \\ &\quad \textcircled{1} \qquad \qquad \qquad \textcircled{2} \end{aligned}$$

$$\begin{aligned} \textcircled{1} &= -\int_{\Omega} \langle \nabla u, \nabla (\varphi \varphi) - \varphi \nabla \varphi \rangle \\ &= -\int_{\Omega} \langle \nabla u, \nabla (\varphi \varphi) \rangle + \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \varphi \\ &\quad \parallel \qquad \qquad \qquad \textcircled{=} \\ &\quad \int_{\Omega} f \varphi \varphi \qquad \qquad \qquad \textcircled{=} \end{aligned}$$

$$\begin{aligned} \textcircled{2} &= -\int_{\Omega} \langle \underbrace{u \nabla \varphi}_{\vec{E}}, \nabla \varphi \rangle = \int_{\Omega} \varphi \operatorname{div} (u \nabla \varphi) \\ &\quad \qquad \qquad \text{div } \vec{E} \\ &= \int_{\Omega} \varphi \langle \nabla u, \nabla \varphi \rangle + \varphi u \Delta \varphi \end{aligned}$$

The sum of $\textcircled{=}$ is $\int_{\Omega} g \varphi$ — — — — —