

ELLIPTIC REGULARITY

$$\Delta u = \sin u$$

DBC in $\partial\Omega$

$$\Delta u = u^{2021}$$

DBC in $\partial\Omega$

We proved that there exists $u \in H^1(\Omega)$ s.t.

1-st int. form of ELE

$$-\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle dx = \int_{\Omega} \sin u \varphi$$

$$-\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle dx = \int_{\Omega} u^{2021} \varphi dx$$

$\forall \varphi \in C_c^\infty(\Omega) \rightsquigarrow$ and therefore also $\forall \varphi \in H_0^1(\Omega)$

ELLIPTIC EQUATION

1.1 Basic linear equation

$$\Delta u = f(x)$$

• weak formulation

$$-\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} f \varphi$$

• Functional

$$F(u) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 + f(x)u \right\} dx$$

$$L(x, s, p) := \frac{1}{2} |p|^2 + f(x)s$$

1.2 Linear equation with coefficient

$$\operatorname{div}(A(x) \nabla u) = f$$

↑
symmetric square matrix with
entries depending on x

• weak formulation

$$-\int_{\Omega} \langle A(x) \nabla u(x), \nabla \varphi(x) \rangle = \int_{\Omega} f \varphi$$

↑
scalar product

- Functional

$$F(u) = \int_{\Omega} \underbrace{\langle A(x) \nabla u(x), \nabla u(x) \rangle + f u}_{\text{quadratic form}} dx$$

$$L(x, s, p) = \langle A(x) p, p \rangle + f(x) s$$

Def. The matrix $A(x)$ is uniformly elliptic in Ω if there exists $\nu > 0$ s.t.

$$\langle A(x) p, p \rangle \geq \nu \|p\|^2 \quad \forall p \in \mathbb{R}^d \quad \forall x \in \Omega$$

The KEY POINT is that

$$\nu \int_{\Omega} |\nabla u|^2 dx \leq \int_{\Omega} \langle A(x) \nabla u(x), \nabla u(x) \rangle dx$$

With sums

$$\langle A(x) \nabla u(x), \nabla u(x) \rangle = \sum_{i=1}^d \sum_{j=1}^d A_{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)$$

$$\operatorname{div}(A(x) \nabla u(x)) = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sum_{j=1}^d A_{i,j}(x) \frac{\partial u}{\partial x_j}(x) \right)$$

1.3 General linear equation

... $L(x, s, p)$ = general polynomial of degree 2 in the variables s and p with coeff. depending on x .

2 SEMILINEAR EQUATION

$$\operatorname{div}(A(x) \nabla u(x)) = g(x, u) \quad \uparrow \uparrow$$

- Weak formulation

$$-\int_{\Omega} \langle A(x) \nabla u(x), \nabla \varphi(x) \rangle dx = \int_{\Omega} g(x, u(x)) \varphi(x) dx$$

- Functional

$$F(u) := \int_{\Omega} \{ \langle A(x) \nabla u, \nabla u \rangle + G(x, u) \} dx$$

$$\frac{\partial G}{\partial s}(x, s) = g(x, s)$$

$$L(x, s, p) = \dots$$

3 QUASILINEAR EQUATION

$$\operatorname{div}(A(x, u, \nabla u) \nabla u) = g(x, u, \nabla u)$$

$\uparrow \uparrow$

The matrix depends
on $x, u, \nabla u$

- Weak formulation : what we expect

- Functional $\int_{\Omega} L(x, u, \nabla u) dx$

4 FULLY NONLINEAR EQUATIONS

$$F(x, u, \nabla u, D^2 u) = 0$$

$\uparrow$
Hessian matrix

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Elliptic regularity

→ INTERNAL

→ BOUNDARY REGULARITY

$$\Delta u(x) = f(x) \quad \text{in } \Omega$$

Internal regularity : $f \in \text{Space}_{\text{loc}}(\Omega) \Rightarrow D^2 u \in \text{Space}_{\text{loc}}(\Omega)$

Boundary regularity : $f \in \text{Space}(\Omega) \Rightarrow D^2 u \in \text{Space}(\Omega)$

Surprise : we assume the sum of pure 2-nd order deriv. in some space, and we obtain ALL 2-nd order deriv. in same space

"If we control $x^2 + y^2 + z^2$ we control $x^2, y^2, z^2, xy, yz, zx$ "

$$f \in H_{\text{loc}}^k(\Omega) \Rightarrow D^2 u \in H_{\text{loc}}^k(\Omega)$$

L^2 -theory

$$f \in W_{\text{loc}}^{k,p}(\Omega) \Rightarrow D^2 u \in W_{\text{loc}}^{k,p}(\Omega)$$

L^p -theory $p \in (1, \infty)$

$$f \in C_{\text{loc}}^{0,\alpha}(\Omega) \Rightarrow D^2 u \in C_{\text{loc}}^{0,\alpha}(\Omega)$$

SCHAUDER THEORY $\alpha \in (0,1)$

The same without "loc"

$$f \in L^\infty \Rightarrow D^2 u \in L^\infty$$

$$f \in L^1 \Rightarrow D^2 u \in L^1$$

$$f \in C^0 \Rightarrow D^2 u \in C^0$$

$$f \in \text{Lip} \Rightarrow D^2 u \in \text{Lip}$$

L^2 theory and L^p theory include the case $k=0$.

$$\Delta u = \underbrace{\sin u}_{f(x) \in L^2} \quad \text{and } \Omega = \text{ball}$$

$$f(x) \in L^2(\Omega) = \mathcal{D}^2 u \in L^2(\Omega) \rightsquigarrow \Delta u \in L^2 \text{ and therefore}$$

$$\downarrow \quad \text{the equation is satisfied}$$

$$u \in H^2(\Omega) \quad \text{in diff. form}$$

$$\Rightarrow \sin u = f(x) \in H^2(\Omega) \Rightarrow u \in H^4(\Omega) \Rightarrow \dots u \in H^k \text{ for every } k \text{ and therefore } u \in C^\infty(\Omega)$$

IT DOES NOT WORK

$$\text{It is false that } u \in H^2 \Rightarrow \sin u \in H^2$$

$$u \in H^1 \Rightarrow \sin u \in H^1 \quad \text{TRUE}$$

d=1 $V = \sin u(x)$

$$V'(x) = \cos u(x) \cdot u'(x)$$

$$V''(x) = \underbrace{-\sin u(x)}_{\substack{\uparrow \\ \text{bounded}}} \cdot \underbrace{[u'(x)]^2}_{\substack{\uparrow \\ \text{a priori not in } L^2, \text{ unless } u' \in L^4}} + \underbrace{\cos u(x)}_{\substack{\uparrow \\ \text{bounded}}} \cdot \underbrace{u''(x)}_{\substack{\uparrow \\ L^2}}$$

If we want the implication $u \in H^2 \Rightarrow \sin u \in H^2$ we need

$$H^1 \rightarrow L^4, \text{ namely } 4 \leq 2^* = \frac{2d}{d-2}$$

$$\rightsquigarrow 4d-8 \leq 2d \rightsquigarrow d \leq 4$$

$$\text{--- } 0 \text{ --- } 0 \text{ ---}$$