

Two variational problems in higher dimension

Let $\Omega \subseteq \mathbb{R}^d$ be a ball (any regular bounded open set is the same)
we want to solve

$$\begin{aligned} \Delta u &= \sin u & \text{in } \Omega \\ \text{DBC} & & \text{on } \partial\Omega \end{aligned}$$

$$\begin{aligned} \Delta u &= u^{2021} & \text{in } \Omega \\ \text{DBC} & & \text{on } \partial\Omega \end{aligned}$$

We expect a variational formulation of the form

$$\begin{aligned} \int_{\Omega} \left\{ \frac{1}{2} |\nabla u(x)|^2 - \cos u(x) \right\} dx \\ + \text{DBC} \end{aligned}$$

$$\begin{aligned} \int_{\Omega} \left\{ \frac{1}{2} |\nabla u(x)|^2 + \frac{1}{2022} u^{2022}(x) \right\} dx \\ + \text{DBC} \end{aligned}$$

Discussion on DBC

Does it make sense to minimize them
with pointwise constraints such as
 $u(\text{origin}) = \text{something}$

Answer: depends on p and d . It makes sense if we have

$$\int_{\Omega} |\nabla u(x)|^p dx \quad \text{with } p > d.$$

For the future: prescribing the values on a subset $E \subseteq \Omega$
makes sense if the p -capacity of E is $\neq 0$.

If we want to prescribe the values of u in $\partial\Omega$ it's enough to say that $\text{Tr } u = \text{given function}$

$\uparrow$
trace

but the given function has to be in the image of the trace, but the image of the trace is strange, for example in $\dim = 2$ it is not true that any continuous function on S^1 is the trace of a function $u \in H^1 = W^{1,2}$ (ball).

Strange answer: $\text{Tr} : W^{1,p}(\Omega) \rightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$

is surjective

$\uparrow$
fractional Sobolev space,
with "fractional derivatives"

What we do in general is to prescribe DBC in the form

$$\text{Tr } u = \text{Tr } u_0$$

$\uparrow$
given
function
in $W^{1,p}(\Omega)$

$$u - u_0 \in W_0^{1,p}(\Omega)$$

Let us start with the direct method.

$$\int_{\Omega} \frac{1}{2} |\nabla u|^2 - \cos u$$

$$\int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{u^{2022}}{2022}$$

Weak formulation in $H^1(\Omega) = W^{1,2}(\Omega)$. So $F : H^1(\Omega) \rightarrow ??$

In the first case real values, in the second case it depends on d : if $2022 < 2^*$, then real values, otherwise $\mathbb{R} \cup \{+\infty\}$

$\frac{2d}{d-2}$ (the only case is $d=2$)

Compactness Let us assume that $F(u_n) \leq M$.

In both cases we obtain

$$\|\nabla u_n\|_{L^2(\Omega)} \leq M' \quad (\text{usual argument})$$

We need a bound on u_n in some space.

In the second case we have

$$\|u_n\|_{L^{2^{**}}(\Omega)} \leq M'' \text{ and therefore } \|u_n\|_{L^2(\Omega)}$$

In the first case we use Poincaré inequality if $u \in W_0^{1,p}(\Omega)$:

$$\|u_n\|_{L^2(\Omega)} \leq \text{const.} \cdot \|\nabla u\|_{L^2(\Omega)}$$

If $u - u_0 \in W_0^{1,p}(\Omega)$, then

$$\begin{aligned} \|u_n\|_{L^2(\Omega)} &\leq \|u_0\|_{L^2(\Omega)} + \|u_n - u_0\|_{L^2(\Omega)} \\ &\leq \|u_0\|_{L^2(\Omega)} + \text{const} \cdot \|\nabla(u_n - u_0)\|_{L^2(\Omega)} \\ &\leq \underbrace{\|u_0\|_{L^2(\Omega)} + \text{const} \cdot \|\nabla u_0\|_{L^2(\Omega)}}_{\text{given}} + \text{const} \cdot \|\nabla u_n\|_{L^2(\Omega)} \end{aligned}$$

In any case from $F(u_n) \leq M$ we get $\|u_n\|_{1,2,\Omega} \leq M'$
FULL NORM in $H^1(\Omega)$

Compact embedding $\Rightarrow u_{n_k} \rightarrow u_\infty$ in $L^2(\Omega)$
 $\uparrow$
more generally in $L^q(\Omega)$ for $q < 2^*$

Up to a further subsequence

$$\nabla u_{n_k} \rightharpoonup \nabla u_\infty \text{ weakly in } L^2(\Omega)$$

LSC In both cases we have that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_{n_k}(x)|^2 dx \geq \int_{\Omega} |\nabla u_{\infty}(x)|^2 dx$$

↑
LSC of the norm
wrt weak conv.

In the first case

$$\lim_{k \rightarrow \infty} \int_{\Omega} \cos(u_{n_k}(x)) dx = \int_{\Omega} \cos(u_{\infty}(x)) dx$$

↑
pointwise conv.
+ trivial domination

[Here we do NOT need a further subsequence, because any subseq. has a further subsubseq. for which convergence is true]

↑ (SUB-SUB lemma)

In the second case we have that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} u_{n_k}^{2022} dx \geq \int_{\Omega} u_{\infty}^{2022} dx$$

↑
FATOU lemma
+ equi-bound from below

Back to compactness: we need to be sure that $\text{Tr } u_{\infty} = \text{Tr } u_0$

This is true because of the continuity of the trace

$$\left. \begin{array}{l} u_n \rightarrow u_{\infty} \text{ in } L^2 \\ \|\nabla u_n\|_{L^2} \text{ bounded} \end{array} \right\} \Rightarrow \text{Tr } u_n \rightarrow \text{Tr } u_{\infty} \text{ in } L^2(\partial\Omega)$$

In this moment a min. exists in $H^1(\Omega)$.

ELE In both cases take a wie. point u , and consider
 $\varphi(t) := F(u + t\sigma)$ with $\sigma \in C_c^\infty(\Omega)$

In the usual way we discover that

$$0 = \varphi'(0) = \int_{\Omega} \langle \nabla u(x), \nabla \sigma(x) \rangle + \sin u(x) \cdot \sigma(x) dx$$

If we **could** integrate by parts ... then

$$0 = \int_{\Omega} (-\Delta u(x) + \sin u(x)) \sigma(x) dx$$

$$\text{FLCV} \Rightarrow \Delta u = \sin u$$

UNFORTUNATELY WE CANNOT

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Traces again let us assume $d = 4$, and $u \in W^{1,p}(\Omega)$

